Isomorphism:

Let V and W be vector spaces. We say that V is isomorphic to W if there exists a linear transformation $T : V \to W$ that is invertible. Such a linear transformation is called an isomorphism from V onto W.

In other words, $T : V \to W$ is an isomorphism from V to W if
(i) T is one-one
(ii) T is onto
(iii) T is linear

Example:

Define $T : \mathbb{R}^2 \to P_1(\mathbb{R})$ by

$$T(a_1, a_2) = a_1 x + a_2$$

then T is an isomorphism, hence $\mathbb{R}^2$ and $P_1(\mathbb{R})$ are isomorphic.

(i) $T(a_1, a_2) = T(b_1, b_2)$

$\Rightarrow a_1 x + a_2 = b_1 x + b_2$

$\Rightarrow a_1 = b_1$ & $a_2 = b_2$

$\Rightarrow (a_1, a_2) = (b_1, b_2)$

(ii) for every $a_1 x + a_2 \in P_1(\mathbb{R})$, $\exists (a_1, a_2) \in \mathbb{R}^2$ such that

$T(a_1, a_2) = a_1 x + a_2$

$\therefore$ T is onto

(iii) Clearly, T is linear (Check!)

Theorem:

Let V and W be finite dimensional vector spaces (over the same field). Then V is isomorphic to W if and only if $\dim(V) = \dim(W)$

Proof:

Suppose V is isomorphic to W

$\Rightarrow \exists\, T: V \to W$ such that T is linear and invertible.

$\Rightarrow \dim(V) = \dim(W)$

Now, suppose that $\dim(V) = \dim(W) = n$ (say)

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_n\}$ be bases for V and W respectively.

There exists a unique linear transformation $T: V \to W$ such that $T(v_i) = w_i$ for all $i = 1, 2, \ldots, n$ — (1)

Now β spans V

$\Rightarrow T(\beta)$ spans $R(T)$

$\Rightarrow \gamma$ spans $R(T)$ (from eqn (1))

$\Rightarrow \text{span}(\gamma) = R(T)$

$\Rightarrow R(T) = W$ $\left(\because \gamma \text{ is a basis for } W \Rightarrow \text{span}(\gamma) = W \right)$

$\therefore T$ is onto

Now, T is also one-one $\left(\because \dim V = \dim W,\ T \text{ is onto} \Leftrightarrow T \text{ is one-one} \right)$

Hence, T is an isomorphism.

Corollary:

Let V be a vector space over F. Then V is isomorphic to F^n if and only if $\dim(V) = n$

Proof:

$\because \dim(F^n) = n$. Apply previous theorem.

Theorem:

Let V and W be finite-dimensional vector spaces over F of dimensions n and m respectively. Let β and γ be ordered bases for V and W respectively. Then the function $\Phi : \mathcal{L}(V,W) \to M_{m\times n}(F)$ defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V,W)$ is an isomorphism.

Proof:

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$ and let A be any $m\times n$ matrix.

Then there exists a unique linear transformation $T: V \to W$ such that

$$T(v_j) = \sum_{i=1}^{m} A_{ij} w_i \quad \text{for } 1 \leq j \leq n$$

$$\Rightarrow [T]_\beta^\gamma = A$$

$$\Rightarrow \Phi(T) = A$$

$\Rightarrow \Phi$ is one-one and onto

Now, $\Phi(aT + U) = [aT+U]_\beta^\gamma = a[T]_\beta^\gamma + [U]_\beta^\gamma$ (here $a \in F$)

$$= a\Phi(T) + \Phi(U)$$

$\therefore \Phi$ is linear, Thus Φ is an isomorphism.

Corollary: Let V and W be finite-dimensional vector spaces of dimensions n and m respectively. Then $\mathcal{L}(V,W)$ is finite dimensional of dimension mn.

Proof: $\because$ $\Phi: \mathcal{L}(V,W) \to M_{m\times n}(F)$ defined by

$\Phi(T) = [T]_\beta^\gamma$ is an isomorphism

$\Rightarrow$ $\mathcal{L}(V,W)$ is isomorphic to $M_{m\times n}(F)$

$\Rightarrow$ dim. $\mathcal{L}(V,W) =$ dim. $(M_{m\times n}(F))$

$\Rightarrow$ dim. $\mathcal{L}(V,W) = mn$

Def^n: Let β be an ordered basis for an n-dimensional vector space V over the field F. The standard representation of V with respect to β is the function $\phi_\beta : V \to F^n$ defined by

$$\phi_\beta(x) = [x]_\beta \quad \text{for each } x \in V$$

Example: Let $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(1,2),(3,4)\}$ be ordered bases for R^2. For $x = (1,-2)$, we have

$$\phi_\beta(x) = [x]_\beta = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\phi_\gamma(x) = [x]_\gamma = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$\because x = (1,-2) = 1.(1,0) + (-2).(0,1)$

and

$x = (1,-2) = (-5).(1,2) + (2).(3,4)$

Theorem:

For any finite-dimensional vector space V with ordered basis β, ϕ_β is an isomorphism.

Proof:

$$\phi_\beta : V \to F^n \quad \text{as} \quad \phi_\beta(x) = [x]_\beta \quad \forall\, x \in V.$$

Clearly, ϕ_β is one-one & onto

and $\phi_\beta(ax+y) = [ax+y]_\beta = a[x]_\beta + [y]_\beta = a\,\phi_\beta(x) + \phi_\beta(y)$

$\therefore$ ϕ_β is linear. Thus ϕ_β is an isomorphism.

If we define $T : V \to W$ such that T is linear transformation. V and W are vector spaces of dimension n and m respectively.

Define $A = [T]_\beta^\gamma$, where β and γ are ordered bases for V and W respectively.

Then we can easily see that

$$L_A \phi_\beta = \phi_\gamma T$$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\
F^n \arrow[r, "L_A"'] & F^m
\end{tikzcd}$$

The Change of Coordinate Matrix:

Theorem: Let β and β' be two ordered bases for a finite dimensional vector space V and let $Q = [I_V]_{\beta'}^{\beta}$. Then

(a) Q is invertible

(b) for any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$

Proof: (a) $\because$ I_V is invertible and $Q = [I_V]_{\beta'}^{\beta}$

$\Rightarrow$ Q is invertible

(b) for any $v \in V$

$$[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^{\beta}[v]_{\beta'} = Q[v]_{\beta'}.$$

Def^n: Let β and β' be two ordered bases for a finite dimensional vector space V. The matrix $Q = [I_V]_{\beta'}^{\beta}$ is called a change of coordinate matrix.

We say that Q changes β'-coordinates into β-coordinates

Observe that, if $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x_1', x_2', \ldots, x_n'\}$, then $x_j' = \sum_{i=1}^{n} Q_{ij} x_i$ for $j = 1, 2, \ldots, n$

Note: If Q changes β'-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into β'-coordinates.

Example: In $\mathbb{R}^2$, let $\beta = \{(1,1),(1,-1)\}$ and $\beta' = \{(2,4),(3,1)\}$.

Let $V = \mathbb{R}^2$.

$$I_V(2,4) = (2,4) = 3(1,1) + (-1)(1,-1)$$

$$I_V(3,1) = (3,1) = 2(1,1) + (1)(1,-1)$$

$$\therefore [I_V]_{\beta'}^{\beta} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

Thus, $[(2,4)]_\beta = Q[(2,4)]_{\beta'} = Q\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\because (2,4) = 1.(2,4) + 0.(3,1)$

$\Rightarrow (2,4)_{\beta'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Similarly $[(3,1)]_\beta = Q[(3,1)]_{\beta'} = Q\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Linear Operator:

A linear transformation that map a vector space V onto ~~that~~ itself is called a linear operator.

If T is a linear operator on a finite dimensional vector space V (i.e. $T: V \to V$). Then V can be represented by the matrices $[T]_\beta$ and $[T]_{\beta'}$. What is the relationship between these two matrices?

Theorem:

Let T be a linear operator on a finite dimensional vector space V and let β and β' be ordered bases ~~for~~ V. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q.$$

Proof:

Let I_V be the identity transformation on V.

Then $Q = [I_V]_{\beta'}^{\beta}$ and $T = I_V T = T I_V$.

Now $$Q[T]_{\beta'} = [I_V]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} = [I_V T]_{\beta'}^{\beta}$$
$$= [T]_{\beta'}^{\beta} = [T I_V]_{\beta'}^{\beta}$$
$$= [T]_{\beta}^{\beta}[I_V]_{\beta'}^{\beta} \qquad \left(\because [UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}\right)$$
$$= [T]_{\beta}Q$$

$$\therefore Q[T]_{\beta'} = [T]_{\beta}Q$$

$$\Rightarrow Q^{-1}Q[T]_{\beta'} = Q^{-1}[T]_{\beta}Q \qquad (\because Q \text{ is invertible})$$

$$\Rightarrow [T]_{\beta'} = Q^{-1}[T]_{\beta}Q.$$

Corollary:

Let $A \in M_{n\times n}(F)$ and let γ be an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n\times n$ matrix whose jth column is the jth vector of γ.

Proof: $L_A : F^n \to F^n$ defined as

$L_A(x) = Ax \quad \forall \; x \in F^n$

standard ~~order~~ Basis for F^n is $\beta = \{e_1, e_2, \ldots, e_n\}$

where $e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \to$ jth entry.

Clearly, $[L_A]_\beta = A$ and $[I_{F^n}]_\gamma^\beta = Q$

using Previous theorem,

$[L_A]_\gamma = Q^{-1}AQ$.

Example: Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a - b \\ a + 3b \end{pmatrix}$$

Let $\beta = \{(1,1), (1,-1)\}$ and $\beta' = \{(2,4), (3,1)\}$ be ~~stan~~ ordered bases for $\mathbb{R}^2$.

then we can easily see that $Q = [I_V]_{\beta'}^{\beta} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$

$[T]_\beta = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$, $[T]_{\beta'} = ?$

$\because Q = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ $\qquad |Q| = 5$

$$Q^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\therefore [T]_{\beta'} = Q^{-1}[T]_{\beta}Q = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow [T]_{\beta'} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 & 7 \\ -6 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 20 & 5 \\ -10 & 10 \end{bmatrix}$$

$$\Rightarrow [T]_{\beta'} = \begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}.$$

Check with alternative method.

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a - b \\ a + 3b \end{pmatrix}$$

$$\beta' = \{(2,4), (3,1)\}$$

$$T\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6-4 \\ 2+12 \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 4 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} 2c_1 + 3c_2 = 2 \\ 4c_1 + c_2 = 14 \end{array}\right\} \Rightarrow c_1 = 4, c_2 = -2.$$

$$\therefore T\begin{pmatrix} 2 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 4 \end{pmatrix} + (-2)\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{—— (1)}$$

Now, $T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 9-1 \\ 3+3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 4 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\Rightarrow \left.\begin{array}{l} 2c_1 + 3c_2 = 8 \\ 4c_1 + c_2 = 6 \end{array}\right\} \Rightarrow c_1 = 1, c_2 = 2$$

$$\therefore T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} 2 \\ 4 \end{pmatrix} + (2)\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{—— (2)}$$

using eqn (1) & (2), $[T]_{\beta'} = \begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}$.

Example: Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}$

and $\gamma = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ be an ordered basis for $\mathbb{R}^3$

Let Q be the 3×3 matrix whose jth column is the jth vector of γ. Find $[L_A]_\gamma$?

Sol:

$$Q = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow Q^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we have

$$[L_A]_\gamma = Q^{-1} A Q = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow [L_A]_\gamma = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 5 & 3 \\ -1 & 3 & 5 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\Rightarrow [L_A]_\gamma = \begin{bmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{bmatrix}.$$

Def^n: (Similar Matrices):

Let A and B be matrices in $M_{n\times n}(F)$. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.

Note: ① The relation of similarity is an equivalence relation. So we can ~~see~~ say that A & B are similar if there exists an invertible matrix Q such that $B = Q^{-1}AQ$

② If T is a linear operator on a finite-dimensional vector space V and if β and β' are any ordered basis for V, then $[T]_{\beta'}$ is similar to $[T]_{\beta}$

$$\left(\because [T]_{\beta'} = Q^{-1}[T]_{\beta}Q\right)$$