Theorem: Aut(G) and Inn(G) are Groups w.r.t to function composition.

proof: Aut (G) consists of $f: G \to G$ (1-1, onto Homo.)

consider set A(G) consists of function $f: G \to G$ where f is 1-1, onto

$\therefore$ Aut (G) $\subseteq$ A(G)

To show: A(G) is group under composition

let $f, g \in A(G)$, 1-1 & onto

then $f \circ g \in A(G)$ ($\because$ f, g are 1-1 & onto)

(ii) $f\circ(g\circ h) = (f\circ g)\circ h$ ($\because$ function composition is associative)

(iii) Let $f \in A(G)$
then $f\circ I(x) = I\circ f(x) = f(x)$
and clearly $I(x) = x$ is 1-1, onto

(iv) let $f \in A(G)$, is 1-1 & onto
then $\exists\ f^{-1} \in A(G)$ which is 1-1 & onto (also)
i.e $f\circ f^{-1} = I$

$\therefore$ $A(G)$ is group w.r.t function composition.

to show: $Aut(G)$ is subgroup of $A(G)$

Let $T_1, T_2 \in Aut(G)$
to show: $T_1 \circ T_2 \in Aut(G)$
Since T_1 & T_2 are 1-1 & onto — (1)
$\therefore$ $T_1\circ T_2$ is also 1-1 & onto — (1)

show: (Homomorphism)

$T_1\circ T_2(ab) = T_1(T_2\, ab)$ where $a, b \in G$
$= T_1(T_2(a)\ T_2(b))$ ($\because$ T_2 is Homo.)
$= T_1(T_2(a))\ T_1(T_2(b))$ ($\because$ T_1 is Homo.)
$= T_1\circ T_2(a)\ T_1\circ T_2(b)$ — (2)

from (1) & (2)
$T_1\circ T_2 \in Aut(G)$ — (3)

Next to show: $T_1^{-1} \in Aut(G)$
Let $T_1 \in Aut(G)$
T_1 is 1-1 & onto $\Rightarrow$ T^{-1} exists & also 1-1 & onto
let $a, b \in G$
Let $T_1^{-1}(a) = a_1$, $T_1^{-1}(b) = b_1$

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Now $T_1^{-1}(a) = a_1$ & $T_1^{-1}(b) = b_1$

$\Rightarrow a = T_1(a_1)$, $b = T_1(b_1)$

$\Rightarrow ab = T_1(a_1)\,T_1(b_1)$

$= T_1(a_1 b_1)$ ($\because T_1 \in Aut(G)$ so Homomorphism)

$\Rightarrow T_1^{-1}(ab) = a_1 b_1$

$= T_1^{-1}(a)\,T_1^{-1}(b)$

$\therefore T_1^{-1} \in Aut(G)$ — (4)

$\therefore$ from (3) & (4)

Aut(G) is subgrp of A(G)

$\therefore$ Aut(G) is group itself

Show: Inn(G) is group

Since every inner automorphism is automorphism

$\therefore Inn(G) \subseteq Aut(G)$

to show: Inn(G) is subgrp of Aut(G)

$Inn(G) = \{T_a \mid a \in G\}$

Let $T_a, T_b \in Inn(G)$

to show: $T_a \circ T_b \in Inn(G)$

Now, $T_a \circ T_b(y) = T_a(byb^{-1})$ where $y \in G$

$= a(byb^{-1})a^{-1}$

$= (ab)y(ab)^{-1}$

$= T_{ab}(y)$

Also, $T_a \circ T_b$ is 1-1 & onto

$\therefore T_a \circ T_b = T_{ab} \in Inn(G)$ — (1)

Let $T_a \in Inn(G)$

As, $a \in G$ (G is group)

$\Rightarrow a^{-1} \in G$

$\Rightarrow T_{a^{-1}} \in Inn(G)$

Now $T_a \circ T_{a^{-1}} = T_{aa^{-1}} = T_e = I$

$T_e(x) = exe^{-1} = x$

$\Rightarrow T_e = I$ (Identity map)

$\therefore (T_a)^{-1} = T_{a^{-1}} \in Inn(G)$ — (2)

$\therefore$ from (1) & (2)

Inn(G) is subgrp of Aut(G)

$\therefore$ Inn(G) is grp itself.

Remark: let $G = \{e, a, b, c, \ldots\}$

then $Inn(G) = \{\phi_e, \phi_a, \phi_b, \phi_c \ldots\}$

However, ϕ_a may be equal to ϕ_b even though $a \neq b$

Question: Find $Inn(D_4)$

Solution: $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$

(1) $R_0 \in D_4$

$T_{R_0}(x) = R_0 x R_0^{-1} = x$

$\Rightarrow T_{R_0}(x) = x$

$\therefore T_{R_0}(R_{90}) = R_0 R_{90} R_0^{-1} = R_{90}$

$T_{R_0}(R_{180}) = R_{180}$

$T_{R_0}(R_{270}) = R_{270}$

$T_{R_0}(H) = H$

$T_{R_0}(V) = V$

$T_{R_0}(D) = D$

$T_{R_0}(D') = D'$

(2) $R_{90} \in D_4$

$T_{R_{90}}(x) = R_{90} x R_{90}^{-1}$

$T_{R_{90}}(R_0) = R_0$

$T_{R_{90}}(R_{90}) = R_{90} R_{90} R_{90}^{-1} = R_{90}$

$T_{R_{90}}(R_{180}) = R_{180}$

$T_{R_{90}}(R_{270}) = \underbrace{R_{90} R_{270}} R_{90}^{-1} = R_0 R_{90}^{-1} = R_{270}$

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$T_{R_{90}}(H) = R_{90} H R_{90}^{-1}$
$= R_{90} H R_{270}$
$= V$

$T_{R_{90}}(V) = R_{90} V R_{90}^{-1} = H$

$T_{R_{90}}(D) = R_{90} D R_{90}^{-1} = D'$

$T_{R_{90}}(D') = R_{90} D' R_{90}^{-1} = D$

(iii) Since $Z(D_4) = \{R_0, R_{180}\}$ (by definition of (centre of group)

$\therefore T_{R_{180}}(x) = R_{180}\, x\, R_{180}^{-1}$

$= x R_{180} R_{180}^{-1}$ (by definition of centre of group)

$= x$ (no rotation means 0°)

$= T_{R_0}(x)$

(iv) $R_{270} \in D_4$ and

$T_{R_{270}}(x) = R_{270}\, x\, R_{270}^{-1}$

then $T_{R_{270}}(R_0) = R_{270} R_0 R_{270}^{-1}$
$= R_0$

$T_{R_{270}}(R_{90}) = R_{270} R_{90} R_{270}^{-1} = R_{90}$

$T_{R_{270}}(R_{180}) = R_{270} R_{180} R_{270}^{-1} = R_{180}$

$T_{R_{270}}(R_{270}) = R_{270}$

$T_{R_{270}}(H) = R_{270} H R_{270}^{-1} = V$

$T_{R_{270}}(V) = H$

$T_{R_{270}}(D') = D'$

$T_{R_{270}}(D) = D$

then $T_{270}(x) = T_{R_{90}}(x)$

(v) $H \in D_4$ s.t

$T_H(x) = HxH^{-1}, \ \forall x \in D_4$

$T_H(R_0) = R_0$

$T_H(R_{90}) = HR_{90}H^{-1} = HR_{90}H$

$= R_{270}$

$T_H(R_{180}) = HR_{180}H^{-1} = R_{180}$

$T_H(R_{270}) = \quad = R_{90}$

$T_H(H) = H$

$T_H(V) = V$

$T_H(D) = D'$

$T_H(D') = D$

(vi) $V \in D_4$

$T_V(R_0) = R_0$

$T_V(R_{90}) = R_{270}$

$T_V(R_{180}) = R_{180}$

$T_V(R_{270}) = R_{90}$

$T_V(H) = VHV^{-1} = H$

$T_V(V) = V$

$T_V(D) = D'$

$T_V(D') = D$

$\therefore T_H(x) = T_V(x)$

(vii) $D \in D_4$

Similarly we get $T_D(x) = T_{D'}(x)$

Hence, $T_{R_{90}}(x)$, $T_{R_0}(x)$, $T_H(x)$, $T_D(x)$ are exactly 4 different Inner Automorphism of D_4.

Exercise: Find $Aut(Z_{10})$