Frobenius Method

This method is named after a German mathematician F.G. Frobenius (1849–1917). His contribution was in mathematics but specially in the theory of $\boxed{\text{matrices \& groups}}$

This method is employed to find the powerseries solⁿ of the diff. eqn.

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{--- (I)}$$

$\boxed{\text{When } x = 0 \text{ is the regular singularity.}}$

& power series solⁿ will be of the form

Procedure :

- $y = x^m \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \right)$ --- (II)

$$\boxed{y = \sum_{k=0}^{\infty} a_k x^{m+k}}$$

where m is $\boxed{\text{real or complex no.}}$

- Substituting the value of y, y' & y'' in (I)
- find the indicial eqn. (or a quadratic eqn.) by equating to zero the coeffs. of the lowest power of x.
- find roots m_1 & m_2 of the indicial eqn.
- find the values of $a_1, a_2, a_3 \ldots$ in terms of a_0 by equating to zero the coeffs. of other power of x.
- The complete solⁿ depends on the nature of roots of the indicial eqn.

F-2

Cases :

(I) When roots m_1 & m_2 are distinct & do not differ by an integer.

(e.g. $m_1 = 1$, $m_2 = \frac{5}{2}$)

$$y_1 = x^{m_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$y_2 = x^{m_2}(a_0 + a_2 x + a_2 x^2 + \cdots)$$

Complete solⁿ is given by $\boxed{y = c_1 y_1 + c_2 y_2}$

(II) When roots are equal i.e. $m_1 = m_2$.

then $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=m_1}$

∴ Complete solⁿ is given by $\boxed{y = c_1 y_1 + c_2\left(\frac{\partial y_1}{\partial m}\right)_{m_1}}$

(III) When m_1 & m_2 are distinct & differ by an integer.

(e.g $m_1 = \frac{1}{2}$ & $m_2 = \frac{9}{2}$ or $m_1 = 0$, $m_2 = 4$)

In some cases $\boxed{m_1 < m_2}$.

If some of coeffs. of y series becomes ∞ when $m = m_1$, we modif the form of y replacing a_0 by $b(m - m_1)$. Then the complete solⁿ is given by

$$\boxed{y = c_1 y_{m_2} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}} \text{ or } c_1 y_2 + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

(coeff. becom ∞ when m = m2)

(IV) ~~Roots are distinct + differing by an integer, making some coeffs. indeterminate~~

Otherwise

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

$\boxed{m_1 \neq m_2 \text{ \& not differ by integer}}$

(I) Solve $4xy'' + 2y' + y = 0$; ($x=0$ is a regular singular point)

Sol^n:- Given $4xy''+2y'+y=0$ --- (A)

Dividing by $4x$, we get

$$y'' + \frac{2}{4x}y' + \frac{1}{4x}\cdot y = 0 \quad \text{--- (1)}$$

Compare with $y'' + p(x)y' + Q(x)y = 0$ --- (2)

we get $xp(x) = \frac{2}{4} = \frac{1}{2} \neq \infty$ at $x=0$

& $(x-0)^2 Q(x) = 0 \neq \infty$ at $x=0$

so $xp(x)$ & $(x-0)^2 Q(x)$ are analytic at $x=0$.

So $x=0$ is a regular singular point.

Let the sol^n be of the form.

$$y = [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots] x^m = \sum a_k x^{m+k} \quad \text{--- (3)}$$

$$y' = \sum a_k (m+k) x^{m+k-1} \quad \text{--- (4)}$$

$$y'' = \sum a_k (m+k)(m+k-1) x^{m+k-2} \quad \text{--- (5)}$$

putting (3) (4) & (5) in (A) we get

$$4x \sum a_k (m+k)(m+k-1) x^{m+k-2} + 2\sum a_k (m+k) x^{m+k-1} + \sum a_k x^{m+k} = 0$$

$$\Rightarrow 4 \sum a_k (m+k)(m+k-1) x^{m+k-1} + 2\sum a_k (m+k) x^{m+k-1} + \sum a_k x^{m+k} = 0$$

$$\Rightarrow \sum [4a_k (m+k)(m+k-1) + 2a_k (m+k)] x^{m+k-1} + \sum a_k x^{m+k} = 0 \quad \text{--- (6)}$$

The coeff. of the lowest degree term x^{m-1} in the identity (6) can be obtained by putting $k=0$ in the 1st summation & equating it to zero. The indicial eq^n is.

$$4a_0(m)(m-1) + 2a_0(m) = 0$$

$$2a_0 m[1 + 2(m-1)] = 0$$

ie $m=0$ & $m=\frac{1}{2}$

E-4

The coeff. of next lowest degree term x^m in the identity (6) is obtained by putting $k=1$ in the Ist summation & $k=0$ in the 2nd summation & equating it to zero

$$4a_1 m(m+1) + 2a_1(m+1) + a_0 = 0$$

$$\cancel{2a_1(m+1)[2(m+1)+1] + a_0 = 0}$$

$$2a_1(m+1)[2m+1] + a_0 = 0$$

$$\boxed{a_1 = -\frac{a_0}{2(m+1)(2m+1)}}$$

Now equating to zero the coeff. of x^{m+k} in (6) we get

$$4a_{k+1}(m+k+1)(m+k) + 2a_{k+1}(m+k+1) + a_k = 0$$

$$2a_{k+1}(m+k+1)[2(m+k)+1] = -a_k$$

$$\boxed{a_{k+1} = -\frac{a_k}{2(m+k+1)[1+2(m+k)]}}$$ This is recurrence relation.

When

$k=0$, $\quad a_1 = -\dfrac{a_0}{2(m+1)(1+2m)}$

$k=1$, $\quad a_2 = -\dfrac{a_1}{2(m+2)(2m+3)} = +\dfrac{a_0}{2(m+1)(1+2m)\,2(m+2)(2m+3)}$

$k=2$, $\quad a_3 = -\dfrac{a_2}{2(m+3)(2m+5)} = -\dfrac{a_0}{2(m+3)(2m+5)\,2(m+1)(1+2m)\,2(m+2)(2m+3)}$

$\vdots$

For $m = 0$

$$a_1 = -\frac{a_0}{2} = -\frac{a_0}{2!}.$$

$$a_2 = \frac{a_0}{2\cdot 1\cdot 4\cdot 3} = \frac{a_0}{4!}.$$

$$a_3 = -\frac{a_0}{2\cdot 3\cdot 5\cdot 2\cdot 1\cdot 4\cdot 3} = -\frac{a_0}{6!}.$$

$\vdots$

For $m = \frac{1}{2}$

$$a_1 = -\frac{a_0}{2\cdot\frac{3}{2}\cdot 2} = \frac{-a_0}{2\cdot 3} = \frac{-a_0}{3!}.$$

$$a_2 = +\frac{a_0}{2\cdot 3\cdot \cancel{2}\cdot\frac{5}{2}\cdot 4} = \frac{a_0}{5!}.$$

$$a_3 = -\frac{a_0}{2\cdot 3\cdot 4\cdot 5\cdot \cancel{2}\cdot\frac{7}{2}\cdot 6} = -\frac{a_0}{7!}.$$

$\vdots$

~~Hence the complete sol^n is given by~~ $y = C_1 y_{m=0} + C_2 y_{m=\frac{1}{2}}$

$\cancel{y = C_1 y_1 + C_2 y_2}$

Thus for $m=0$, $y_{m=0} = y_1 = x^m(a_0 + a_1 x + a_2 x^2 + \cdots)\Big|_{m=0}$

$$= a_0\left[1 - \frac{1}{2!}x + \frac{1}{4!}x^2 + \cdots\right]$$

$$= a_0\left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} + \cdots\right]$$

$$= a_0 \cos\sqrt{x}$$

Likewise for $m = \frac{1}{2}$, $y_{m=\frac{1}{2}} = y_2 = x^m(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$

$$y_2 = x^{\frac{1}{2}} a_0\left[1 - \frac{x}{3!} + \frac{x^2}{5!} + \cdots\right]$$

$$= a_0\left[\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} + \cdots\right]$$

$$= a_0 \sin\sqrt{x}$$

Hence complete sol^n is

$$y = C_1 y_1 + C_2 y_2 = C_1 a_0 \cos\sqrt{x} + C_2 a_0 \sin\sqrt{x}$$

$$= c_1' \cos\sqrt{x} + c_2' \sin\sqrt{x}$$ Ans.

Roots equal $\boxed{m_1 = m_2}$

(II) $(x-x^2)y'' + (1-5x)y' - 4y = 0$

Sol^n

$$(x-x^2)y'' + (1-5x)y' - 4y = 0 \quad \text{---(A)}$$

Dividing by $x-x^2$, we get

$$y'' + \left(\frac{1-5x}{x-x^2}\right)y' - \frac{4}{(x-x^2)}y = 0 \quad \text{---(1)}$$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$ --- (2)

we get $xP(x) = \frac{1-5x}{1-x} = 1 \neq \infty$ at $x=0$

$$(x-0)^2 Q(x) = \frac{-4x}{1-x} = 0 \neq \infty \text{ at } x=0$$

Since $xP(x)$ & $(x-0)^2Q(x)$ are analytic at $x=0$.

So $x=0$ is a regular singular point.

Let the sol^n of (A) be of the form

$$y = x^m(a_0 + a_1x + a_2x^2 + \dots a_k x^k) = \sum a_k x^{m+k} \quad \text{---(3)}$$

$$y' = \sum a_k(m+k)x^{m+k-1} \quad \text{---(4)}$$

$$y'' = \sum a_k(m+k)(m+k-1)x^{m+k-2} \quad \text{---(5)}$$

putting (3), (4) & (5) in (A) we get

$$(x-x^2)\sum a_k(m+k)(m+k-1)x^{m+k-2} + (1-5x)\sum a_k(m+k)x^{m+k-1} - 4\sum a_k x^{m+k} = 0$$

$$\Rightarrow \sum a_k(m+k)(m+k-1)x^{m+k-1} - \sum a_k(m+k)(m+k-1)x^{m+k} + \sum a_k(m+k)x^{m+k-1} - \sum 5a_k(m+k)x^{m+k} - \sum 4a_k x^{m+k} = 0$$

$$\Rightarrow \sum[a_k(m+k)(m+k-1) + a_k(m+k)]x^{m+k-1} - \sum[a_k(m+k)(m+k-1) + 5a_k(m+k) + 4a_k]x^{m+k} = 0$$

$$\Rightarrow \sum a_k(m+k)^2 x^{m+k-1} - \sum[a_k(k+m)^2 + 4(k+m) + 4]x^{m+k} = 0$$

$$\Rightarrow \sum a_k (m+k)^2 x^{m+k-1} - \sum a_k (m+k+2)^2 x^{m+k} = 0 \quad \text{--- (6)}$$

which is an identity.

Equating to zero coeff. of the smallest power of x;

i.e x^{m-1}. [put $k=0$ in the first summation & equating it to zero]

The indicial eqn. is

$$a_0 m^2 = 0 \Rightarrow m^2 = 0 \quad , a_0 \neq 0$$

Thus $\boxed{m = 0, 0}$ equal roots

Now equating to zero coeff. of x^{m+k} in (6) we get

$$a_{k+1}(m+k+1)^2 - a_k (m+k+2)^2 = 0$$

$$\boxed{a_{k+1} = + a_k \cdot \left(\frac{m+k+2}{m+k+1}\right)^2} \quad \text{--- (7)}$$

This recurrence relation

if $k=0$, $\quad a_1 = a_0 \left(\frac{m+2}{m+1}\right)^2$

$k=1$, $\quad a_2 = a_1 \left(\frac{m+3}{m+2}\right)^2 = a_0 \left(\frac{m+2}{m+1}\right)^2 \cdot \left(\frac{m+3}{m+2}\right)^2 = \frac{(m+3)^2}{(m+1)^2}$

$k=2$, $\quad a_3 = a_2 \left(\frac{m+4}{m+3}\right)^2 = a_0 \frac{(m+4)^2}{(m+1)^2}$

$k=3$, $\quad a_4 = a_3 \left(\frac{m+5}{m+4}\right)^2 = a_0 \frac{(m+5)^2}{(m+1)^2}$

$k=4$, $\quad a_5 = a_4 \left(\frac{m+6}{m+5}\right)^2 = a_0 \frac{(m+6)^2}{(m+1)^2}$

,
,
,
,

$$\therefore y_1 = x^m a_0\left[1 + \frac{(m+2)^2}{(m+1)^2}x + \frac{(m+3)^2}{(m+1)^2}x^2 + \frac{(m+4)^2}{(m+1)^2}x^3 + \cdots\right] \quad \text{--- (8)}$$

put $m = 0$ in (8) we get

$$y_1 = a_0\left[1 + 2^2 \cdot x + 3^2 \cdot x^2 + 4^2 x^3 + \cdots\right] \quad \text{--- (9)}$$

Now second sol^n can be found by

$$y_2 = \left(\frac{\partial y_1}{\partial m}\right)_{m=0}$$

Differentiating (8) w.r.t 'm'

$$y_2 = \left(\frac{\partial y}{\partial m}\right) = a_0 x^m \log x\left[1 + \frac{(m+2)^2}{(m+1)^2}x + \frac{(m+3)^2}{(m+1)^2}x^2 + \frac{(m+4)^2}{(m+1)^2}x^3 + \cdots\right]$$

$$+ a_0 x^m\left[0 + \frac{(m+1)^2 2(m+2) - (m+2)^2 2(m+1)}{(m+1)^4} \cdot x + \frac{(m+1)^2 2(m+3) - (m+3)^2 2(m+1)}{(m+1)^4} x^2 + \cdots\right]$$

$$\therefore y_2 = \frac{\partial y}{\partial m} = a_0 \log x\left[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right] + a_0\left[0 + \frac{(4-8)}{1}x + \frac{(6-18)}{1} \cdot x^2 + \cdots\right]$$

$$= a_0 \log x\left[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right] - a_0\left[4x + 12x^2 + \cdots\right] \quad \text{--- (10)}$$

$\therefore$ Complete sol^n is $y = C_1 y_1 + C_2 y_2$ put (9) & (10)