

G.E-2

Orthogonality

Defⁿ :- A set $S = \{v_1, v_2, \dots, v_k\}$ of k distinct vectors of $\mathbb{R}^n$ is said to be an orthogonal set of vectors if and only if every pair of distinct vectors in this set is orthogonal i.e. $v_i \cdot v_j = 0 \quad \forall i \neq j$ } i.e dot product of any two vectors is zero

Ex $S = \{(1, 0, -1), (1, \sqrt{2}, 1), (1, -\sqrt{2}, 1)\}$

$$(1, 0, -1) \cdot (1, \sqrt{2}, 1) = 1 + 0 - 1 = 0$$

$$(1, 0, -1) \cdot (1, -\sqrt{2}, 1) = 1 + 0 - 1 = 0$$

$$(1, \sqrt{2}, 1) \cdot (1, -\sqrt{2}, 1) = 1 - 2 + 1 = 0$$

$\Rightarrow S$ is orthogonal set.

Defⁿ :- Orthonormal Vector

A set $S = \{v_1, v_2, \dots, v_k\}$ of k distinct vectors of $\mathbb{R}^n$ is said to be an orthonormal set of vectors if and only if it is orthogonal set and all vectors are unit vectors

Theorem :- orthogonality $\Rightarrow$ linear independence

Let $T = \{v_1, v_2, \dots, v_k\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Then T is linearly independent.

Proof:- let $a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0$
for some $a_1, a_2, \dots, a_k \in \mathbb{R}$

Claim:- $a_1 = a_2 = \dots = a_k = 0$

We know

$$(a_1 v_1 + a_2 v_2 + \dots + a_k v_k) \cdot v_i = 0 \quad \left\{ \begin{array}{l} \text{any } v_i \\ \text{it is orthogonal} \end{array} \right.$$

$$\Rightarrow a_1 (v_1 \cdot v_i) + a_2 (v_2 \cdot v_i) + \dots + a_k (v_k \cdot v_i) = 0$$

$$\Rightarrow \cancel{a_i} + a_i (v_i \cdot v_i) = 0$$

$$\Rightarrow a_i = 0$$

$\left. \begin{array}{l} \because v_i \cdot v_i \neq 0 \\ \text{as } v_i \text{ are} \\ \text{non-zero vectors} \end{array} \right\}$

Defⁿ:- Orthogonal and orthonormal Basis

A Basis B for a subspace W of $\mathbb{R}^n$ is said to be an orthogonal basis for W if and only if B is an orthogonal set.

B is orthonormal set if and only if B is orthonormal Basis for W

Corollary:- If B is an orthonormal set of n non-zero vectors in $\mathbb{R}^n$, then B is an orthogonal Basis for $\mathbb{R}^n$. Similarly, if B is an orthonormal set of n vectors in $\mathbb{R}^n$, then B is an orthonormal Basis for $\mathbb{R}^n$.

Proof:- By above theorem
orthogonal set $\Rightarrow$ linearly independent

$\Rightarrow B$ is Basis for $\mathbb{R}^n$

By if B is orthonormal
so it is orthogonal $\Rightarrow$ linearly independent

$\Rightarrow B$ is Basis for $\mathbb{R}^n$.

Theorem :- Suppose that $B = \{v_1, v_2, \dots, v_n\}$ is a (3) non-empty ordered orthogonal basis for $\mathbb{R}^n$ and v be any vector in $\mathbb{R}^n$. Then

$$v = \frac{v \cdot v_1}{\|v_1\|^2} v_1 + \frac{v \cdot v_2}{\|v_2\|^2} v_2 + \dots + \frac{v \cdot v_n}{\|v_n\|^2} v_n$$

and $a_1, a_2, \dots, a_n$ the coordinatization of v with respect to B is

$$\begin{bmatrix} v \\ B \end{bmatrix} = \begin{bmatrix} \frac{v \cdot v_1}{\|v_1\|^2} \\ \frac{v \cdot v_2}{\|v_2\|^2} \\ \vdots \\ \frac{v \cdot v_n}{\|v_n\|^2} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Proof :- $\because B = \{v_1, \dots, v_n\}$ is an ordered basis for $\mathbb{R}^n$ & $v \in \mathbb{R}^n \Rightarrow$ there exist $a_1, \dots, a_n$ scalars such that

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

} definition of basis and linear combination

Claim :- $a_i = \frac{v \cdot v_i}{\|v_i\|^2}$

$$\begin{aligned} v \cdot v_i &= (a_1 v_1 + a_2 v_2 + \dots + a_i v_i + \dots + a_n v_n) \cdot v_i \\ &= a_1 (v_1 \cdot v_i) + a_2 (v_2 \cdot v_i) + \dots + a_i (v_i \cdot v_i) + \dots + a_n (v_n \cdot v_i) \end{aligned}$$

$\because B$ is orthogonal basis

$$\Rightarrow v_i \cdot v_j = 0 \text{ for } i \neq j$$

$$\Rightarrow v \cdot v_i = a_1(v) + a_2(v) + \dots + a_i(\|v_i\|^2) + \dots + a_n(v) \quad (4)$$

$$\Rightarrow a_i = \frac{v \cdot v_i}{\|v_i\|^2}$$

$$\left\{ \begin{array}{l} v_i \cdot v_i = \|v_i\|^2 \end{array} \right.$$

Corollary :- Suppose $B = \{v_1, \dots, v_n\}$ is an ordered orthonormal basis for $\mathbb{R}^n$, and v be any

vector in $\mathbb{R}^n$. Then

$$v = (v \cdot v_1) v_1 + \dots + (v \cdot v_n) v_n$$

Proof :- by above theorem

$$v = \frac{v \cdot v_1}{\|v_1\|^2} v_1 + \dots + \frac{v \cdot v_n}{\|v_n\|^2} v_n$$

$\therefore B$ is orthonormal basis $\Rightarrow \|v_i\|^2 = 1$

$$\Rightarrow v = (v \cdot v_1) v_1 + \dots + (v \cdot v_n) v_n$$

hence $[v]_B = \begin{bmatrix} v \cdot v_1 \\ \vdots \\ v \cdot v_n \end{bmatrix}$

Ques :- let $B = \{v_1 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), v_2 = (-\frac{1}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}), v_3 = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})\}$

is an orthonormal basis for $\mathbb{R}^3$ & find $[v]_B$ for

$$v = \{1, 5, 3\}$$

Soln ^{verify} ~~find~~ $\|v_1\|^2 = \|v_2\|^2 = \|v_3\|^2 = 1$

$$\& v_1 \cdot v_2 = v_2 \cdot v_3 = v_1 \cdot v_3 = 0$$

$\Rightarrow$ orthonormal

then use ~~for~~ above corollary

gml

Gram-Schmidt Process :- Let $\{w_1, w_2, \dots, w_k\}$ be linearly independent set of vectors in $\mathbb{R}^n$. The

Gram-Schmidt process constructs a new set $\{v_1, v_2, \dots, v_k\}$ of vectors such that $\{v_1, v_2, \dots, v_k\}$ become an orthogonal basis for span $\{w_1, w_2, \dots, w_k\}$.

* Following are the steps in Gram-Schmidt process

Step ① let $v_1 = w_1$

Step ② let $v_2 = w_2 - \left(\frac{w_2 \cdot v_1}{\|v_1\|^2} \right) v_1 = w_2 - \text{Proj}_{v_1} w_2$

Step ③ let $v_3 = w_3 - \left(\frac{w_3 \cdot v_1}{\|v_1\|^2} \right) v_1 - \left(\frac{w_3 \cdot v_2}{\|v_2\|^2} \right) v_2$

Continue this process upto v_k . We will get the set $\{v_1, v_2, \dots, v_k\}$.

Theorem :- Let $B = \{w_1, w_2, \dots, w_k\}$ be a basis for a subspace W of $\mathbb{R}^n$. Then the set $T = \{v_1, v_2, \dots, v_k\}$ obtained by applying the Gram-Schmidt process to B is an orthogonal basis for W . { by definition of Gram-Schmidt process }

Ex :- let $B = \{(2, 1, 0, -1), (1, 0, 2, -1), (0, -2, 1, 0)\}$ of subset of $\mathbb{R}^4$. verify B is a linearly independent set. find the orthogonal basis for $\mathbb{R}^4$.

B is not orthogonal basis as $(2, 1, 0, -1) \cdot (1, 0, 2, -1) \neq 0$

Prove B is linearly independent (verify)

Now using Gram-Schmidt process

$$\text{let } v_1 = w_1 = (2, 1, 0, -1)$$

$$v_2 = w_2 - \left(\frac{w_2 \cdot v_1}{\|v_1\|^2} \right) v_1$$

$$= (1, 0, 2, -1) - \frac{(1, 0, 2, -1) \cdot (2, 1, 0, -1)}{\|(2, 1, 0, -1)\|^2} (2, 1, 0, -1)$$

$$= (1, 0, 2, -1) - \frac{3}{6} (2, 1, 0, -1)$$

$$= (1, 0, 2, -1) - (1, 1/2, 0, -1/2)$$

$$= (0, -1/2, 2, -1/2)$$

$$\text{If } v_3 = w_3 - \left(\frac{w_3 \cdot v_1}{\|v_1\|^2} \right) v_1 - \left(\frac{w_3 \cdot v_2}{\|v_2\|^2} \right) v_2$$

$$\text{If we can find } v_4 = w_4 - \left(\frac{w_4 \cdot v_1}{\|v_1\|^2} \right) v_1 - \left(\frac{w_4 \cdot v_2}{\|v_2\|^2} \right) v_2 - \left(\frac{w_4 \cdot v_3}{\|v_3\|^2} \right) v_3$$

Theorem :- Let W be a subspace of $\mathbb{R}^n$. Then any orthogonal set of non-zero vectors in W can be enlarged to an orthogonal basis for W.

If for orthonormal

Defⁿ (orthogonal complement)

Let W be a subspace of $\mathbb{R}^n$. the orthogonal complement of W , denoted by $W^\perp$, is the set of all vectors of $\mathbb{R}^n$ that are orthogonal to every vector in W .

$$W^\perp = \{x \in \mathbb{R}^n : x \cdot w = 0 \text{ for all } w \in W\}$$

Theorem :- If W is a subspace of $\mathbb{R}^n$, then $v \in W^\perp$ if and only if v is orthogonal to every vector in a spanning set for W .

Proof :- let $S = \{w_1, w_2, \dots, w_k\}$ be spanning set for W i.e. $W = \text{Span}(S)$

$$\text{let } v \in W^\perp$$

claim :- $v \cdot w_i = 0$ for all $w_i \in \text{Span}(S) = W$

by definition $v \cdot w_i = 0$ for all $w_i \in W$

$\Rightarrow$ in particular $v \cdot w_i = 0$

Conversely

let $v \cdot w_i = 0$ for all $w_i \in \text{Span}(S) = W$

$\Rightarrow$ for any $w \in W$ there exist $a_1, a_2, \dots, a_k$ such that

$$w = a_1 w_1 + a_2 w_2 + \dots + a_k w_k$$

$$\text{Now } v \cdot w = a_1 v \cdot w_1 + a_2 v \cdot w_2 + \dots + a_k v \cdot w_k$$

$$= 0$$

$$\Rightarrow v \in W^\perp$$

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ex let $W = \{ (a, 0, 0) : a \in \mathbb{R} \}$ subspace of $\mathbb{R}^3$ and let W is spanned by the set $\{ (1, 0, 0), (0, 0, 1) \}$ find $W^\perp$

$$\text{let } (x, y, z) \in W^\perp$$

$$\Rightarrow (x, y, z) \cdot (1, 0, 0) = 0$$

$$\& (x, y, z) \cdot (0, 0, 1) = 0$$

$$\Rightarrow x = 0, y = 0$$

$$\Rightarrow W^\perp = \{ (0, y, 0) : y \in \mathbb{R} \} = \text{span} \{ (0, 1, 0) \}$$

Theorem :- Let W be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is a subspace of $\mathbb{R}^n$ and $W \cap W^\perp = \{0\}$.

Proof :- clearly $0 \in W^\perp$ because 0 is orthogonal to each vector

$$\text{let } x_1, x_2 \in W^\perp, \text{ and } w \in W$$

$$(x_1 + x_2) \cdot w = x_1 \cdot w + x_2 \cdot w = 0$$

$$\Rightarrow x_1 + x_2 \in W^\perp$$

$$\text{now } (cx) \cdot w = c(x \cdot w) = 0$$

$$\Rightarrow cx \in W^\perp$$

$$\Rightarrow W^\perp \text{ is subspace of } \mathbb{R}^n$$

$$\text{now let } w \in W \cap W^\perp$$

$$\Rightarrow w \in W \& w \in W^\perp$$

$$\Rightarrow w \cdot w = 0$$

$$\Rightarrow w = 0$$

$$\Rightarrow W \cap W^\perp = \{0\}$$