UNIT 4 LINEAR PROGRAMMING - SIMPLEX METHOD

Objectives

After studying this unit, you should be able to :

- describe the principle of simplex method
- discuss the simplex computation
- explain two phase and M-method of computation
- work out the sensitivity analysis
- formulate the dual linear programming problem and analyse the dual variables.

Structure

4.1 INTRODUCTION

Although the graphical method of solving linear programming problem is an invaluable aid to understand its basic structure, the method is of limited application in industrial problems as the number of variables occurring there is substantially large. A more general method known as **Simplex Method** is suitable for solving linear programming problems with a larger number of variables. The method through an iterative process progressively approaches and ultimately reaches to the maximum.or minimum value of the objective function. The method also helps the decision maker to identify the redundant constraints, an unbounded solution, multiple solution and an infeasible problem.

In industrial applications of linear programming, the coefficients of the objective function and the right hand side of the constraints are seldom known with complete certainty. In many problems the uncertainty is so great that the effect of inaccurate coefficients can be predominant. The effect of changes in the coefficients in the maximum.or minimum value of the objective function can be studied through a technique known as **Sensitivity Analysis.**

Every **linear programming problem** has a **dual problem** associated with it. The solution of this problem is readily obtained from the solution of the original problem if simplex method is used for this purpose. The variables of dual problem are known as **dual variables** or **shadow price** of the. various resources. The solution of the dual problem can be used by the decision maker for augmenting the resources.

The methodological aspects of the Simplex method is explained with a linear programming problem with two decision variables in the next section.

4.2 PRINCIPLE OF SIMPLEX METHOD

We explain the principle of the **Simplex method with** the help of the two variable linear programming problem introduced **in** Unit 3, Section 2.

Example I

Maximise $50x_1 + 60x_2$

Subject to :

$$2x_1 + x_2 \leq 300$$
$$3x_1 + 4x_2 \leq 509$$
$$4x_1 + 7x_2 \leq 812$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

Solution

We introduce variables x_3 .>. 0, x_4 0, x5 r 0 **So that the constraints become equations**

$$\begin{aligned} 2x_1 + x_2 + x_3 \quad\quad &= 300 \\ 3x_1 + 4x_2 \quad + x_4 \quad &= 509 \\ 4x_1 + 7x_2 \quad\quad + x_5 &= 812 \end{aligned}$$

The variables **x_3, x_4, x_5** are **known as slack variables** corresponding to the three constraints. The system of **equations** has five variables (including the slack variables) and three equations.

Basic Solution

In the system of equations as presented above we may equate any two variables to zero. The system then consists of three equations with three variables. If this system of three equations with three variables is solvable such a solution is known as a **basic solution.**

In the example considered above suppose we take x, = 0, x_2 = O. The solution of the system with remaining three variables is $x_3 = 300$, $x_4 = 509$, $x_5 = 812$. This is a **basic solution** of the system. The variables x_3, x_4 and x_5 are known as **basic variables** while the variables x_1, x_2 are known as **non basic variables** (variables which are equated to zero).

Since there are three equations and five variables the two non basic variables can be chosen in $5c_2$, = 10 ways. Thus, the maximum number of basic solutions is 10, for in some cases the three variable three equation problem may not be solvable.

In the general case, if the number of constraints of the linear programming problem is m and the number of variables (including the slack variables) is n then there are **at most** ${}^{n}C_{n-m} = {}^{n}C_{m}$ basic solutions.

Basic Feasible Solution

A basic solution of a linear programming problem is a **basic feasible solution** if it is feasible, i.e. all the variables are non negative. The solution $x_3 = 300$, $x_4 = 509$, $x_5 = 812$ is a basic feasible solution of the problem. Again, if the number of constraints is m and the number of variables (including the slack variables) is n, the **maximum** number of basic feasible' solution is ${}^{n}C_{n-m} = {}^{n}C_{m}$

The following result (Hadley, 1969) will help you to identify the extreme points of the convex set of feasible solutions analytically.

Every **basic feasible solution** of the problem is an **extreme point** of the convex set of feasible solutions and every extreme point is a basic feasible solution of the set of Constraints.

When several variables are present in a linear programming problem it is not possible to identify the extreme points geometrically. But we can identify them through the basic feasible solutions. Since one of the basic feasible solutions will maximise or minimise the objective function, we can carry out this search starting from one basic feasible solution to another. The **simplex method** provides a systematic search so that the objective function increases (in the case of maximisation) progressively until the basic feasible solution has been identified where the objective function is maximised. The computational aspect of the simplex method is presented in the next section.

Activity 1

Fill up the blanks :

i) ....variables are introduced to make …………… .... type inequalities equations.

ii) A system with m equations and n variables has at most ……………basic solutions.

iii) A basic solution with m equations and n variables has .... variables equal to zero.

iv) A basic feasible solution is a basic solution whose variables are ....

v) The maximum number of basic feasible solutions in a system with m equations and n variables is ………………..

vi) In a linear programming problem every .... point of the Convex set of feasible solutions is a ....solution of the problem.

vii) The objective function of a linear programming problem is maximised or minimised at a…………………solution.

4.3 COMPUTATIONAL ASPECT OF SIMPLEX METHOD

We again consider the linear programming problem

Maximise $50x_1 + 60x_2$

Subject to :

$$2x_1 + x_2 + x_3 = 300$$
$$3x_1 + 4x_2 + x_4 = 509$$
$$4x_1 + 7x_2 + x_5 = 812$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

The slack variables provide a basic feasible solution to start the simplex computation. This is also known as **initial basic feasible solution.** If z denote the profit then z = 0 corresponding to this basic feasible solution. We denote by C_B the coefficient of the basic variables in the objective function and by X_{13} the numerical values of the basic variables. The numerical values of the basic variables are

$X_{B1} = 300$, $X_{B2} = 509$, $X_{B3} = 812$. The profit $z = 50x_1 + 60x_2$ can be also expressed as $z - 50x_1 - 60x_2 = 0$. The computation starts with the first simplex Table as indicated below :

Table 1

C_B	Basic Variables	C_j / X_B	50 / x_1	60 / x_2	0 / x_3	0 / x_4	0 / x_5
0	x_3	300	2	1	1	0	0
0	x_4	509	3	4	0	1	0
0	x_5	812	4	7	0	0	1
	z		−50	−60	0	0	0

The coefficients of the basic variables in the objective function are $C_{BI} = C_{B2} = CB3 = 0$ The topmost row of Table 1 indicates the coefficient of the variables x_i, x2; x_3, x4 and x_5 in the objective function respectively. The column under x_i presents the coefficient of x_i in the three equations. The remaining columns have also been formed in a similar manner.

On examining the profit equation $z = 50x_1 + 60x_2$ you may observe that if either x_1

or x_2 which is currently non basic is included as a basic variable the profit will increase. Since the coefficient of x_2 is numerically higher we choose x_2 to be included as a basic variable in the next iteration. An equivalent criterion of choosing a new basic variable can be obtained from the last row of Table 1 (corresponding to z). Since the entry corresponding to x_2 is **smaller between the two negative values** x_2 will be included as a basic variable in the next iteration. However with three constraints there can only be three basic variables. Thus by making x_2 a basic variable one of the existing basic variables will become non basic. You may identify this variable using the following line of argument.

From the first equation

$$2x_1 + x_3 = 300 - x_2$$

But $x_1 = 0$. Hence, in order that $x_3 \geq 0$

$$300 - x_2 \geq 0 \quad \text{i.e.} \quad x_2 \leq 300$$

Similar computation from the second and the third equation lead to

$$x_2 \leq \frac{509}{4}, \qquad x_2 \leq \frac{812}{7} = 116$$

Thus $x_2 = \text{Min}\left(\frac{300}{1}, \frac{509}{4}, \frac{812}{7}\right) = 116$

If $x_2 = 116$, from the third equation you may observe that

$$7x_2 + x_5 = 812 \quad \text{i.e.} \quad x_5 = 0$$

Thus the variable x_5 becomes non basic in the next iteration. The revised values of the other two basic variables are

$$x_3 = 300 - x_2 = 184$$
$$x_4 = 509 - 4 \times 116 = 45$$

Referring back to Table 1, we obtain the elements of the next Table (Table 2) using the following rules :

1) In the z row we locate the quantities which are negative. If all the quantities are positive, the inclusion of any non basic variable will not increase the value of the objective function. Hence the present solution maximises the objective function. If there are more than one negative values we choose the variable as a basic variable corresponding to which the z value is least as this is likely to increase the profit most.

2) Let x_i be the incoming basic variable and the corresponding elements of the jth column be denoted by ylj, Y2j and Y3j. If the present values of basic variables are **XB1, XB2** and x_{B3} respectively, then we compute

$$\text{Min}\left[\frac{x_{B1}}{y_{1j}}, \frac{x_{B2}}{y_{2j}}, \frac{x_{B3}}{y_{3j}}\right]$$

for $y_{1j} > 0$, $y_{2j} > 0$, $y_{3j} > 0$. You may note that if any $y_{ij} \leq 0$, this need not be included in the comparison. If the minimum occurs corresponding to $\frac{x_{Br}}{y_{rj}}$ then the rth basic variable will become non basic in the next iteration.

3) Table 2 is computed from Table 1 using the following rules.

   a) The revised basic variables are x_3, x_4 and x_2. Accordingly, we make $C_{B1} = 0$, $C_{B2} = 0$ and $C_{B3} = 60$.

   b) As x_2 is the incoming basic variable we make the coefficient of x_2 one by dividing each element of row 3 by 7. Thus the numerical value of the element corresponding to x_1 is $\frac{4}{7}$, corresponding to x_5 is $\frac{1}{7}$ in Table 2.

   c) The incoming basic variable should appear only in the third row. So we multiply **the third row of Table 2** by 1 and subtract it from the **first row of Table 1 element by element.** Thus the element corresponding to x_2 in the first row of Table 2 is zero. The element corresponding to x_1 is

$$2 - 1 \times \frac{4}{7} = \frac{10}{7}$$

The element corresponding to x_5 is

$$0 - 1 \times \frac{1}{7} = -\frac{1}{7}$$

In this way we obtain the elements of the first and the second row in Table 2. The numerical values of the basic variables in Table 2 can also be computed in a similar manner.

Let C_{B1}, C_{B2}, C_{B3} be the coefficients of the basic variables in the objective function. For example in Table 2 $C_{B1} = 0, C_{B2} = 0, C_{B3} = 60$. Suppose corresponding to a variable j, the quantity z_j is defined as $z_j = C_{B1}, Y_1 + C_{B2}, y_{2j} + C_{B3}\, y_{3j}$. Then the final row (z-row) can also be expressed as $z_j - C_j$. For example

$$z_1 - c_1 = \frac{10}{7} \times 0 + \frac{5}{7} \times 0 + 60 \times \frac{4}{7} - 50 = -\frac{100}{7}$$

$$z_5 - c_5 = -\frac{1}{7} \times 0 - \frac{4}{7} \times 0 + \frac{1}{7} \times 60 - 0 = \frac{60}{7}$$

1) We now apply rule 1 to Table 2. The only negative $z_j - c_j$ is $z_1 - c_1 = -\frac{100}{7}$ Hence x_1 should be made a basic variable at the next iteration.

2) We compute the minimum of the ratios

$$\text{Min}\left[\frac{184}{\frac{10}{7}}, \frac{45}{\frac{5}{7}}, \frac{116}{\frac{4}{7}}\right]$$

$$= \text{Min}\left[\frac{644}{5}, 63, 203\right] = 63.$$

Since this minimum occurs corresponding to x_4, it becomes a non basic variable in next iteration.

3) Table 3 is computed from Table 2 using the rules (a), (b) and (c) as described before.

Table 2

C_B	Basic Variables	C_j X_B	50 x_1	60 x_2	0 x_3	0 x_4	0 x_5
0	x_3	184	$\frac{10}{7}$	0	1	0	$-\frac{1}{7}$
0	x_4	45	$\frac{5}{7}$	0	0	1	$-\frac{4}{7}$
60	x_2	116	$\frac{4}{7}$	1	0	0	$\frac{1}{7}$
	$z_j - c_j$		$-\frac{100}{7}$	0	0	0	$\frac{60}{7}$

Table 3

C_B	Basic Variables	C_j X_B	50 x_1	60 x_2	0 x_3	0 x_4	0 x_5
0	x_3	94	0	0	1	−2	1
50	x_1	63	1	0	0	$\frac{7}{5}$	−4/5
60	x_2	80	0	1	0	−4/5	3/5
	$z_j - c_j$		0	0	0	22	−4

1) $z_5 - c_5 < 0$. Hence x_5 should be made a basic variable in the next iteration.

2) We compute the minimum of the ratios

$$\text{Min}\left[\frac{94}{1}, \frac{80}{3/5}\right] = 94$$

Note that since $y_{25} < 0$, the corresponding ratio is not taken for comparison. The variable x_3 becomes non basic at the next iteration.

3) Table 4 is computed from Table following the usual steps.

Table 4

C_B	Basic Variables	C_j X_B	50 x_1	60 x_2	0 x_3	0 x_4	0 x_5
0	x_5	94	0	0	1	−2	1
50	x_1	691/5	1	0	4/5	−1/5	0
60	x_2	118/5	0	1	−3/5	2/5	0
	$z_j - C_j$		0	0	4	14	0

Since $z_j - C_j \geq 0$ for all j, the objective function cannot be improved any further. Hence the objective function is maximised for $x_1 = \frac{691}{5}$ and $x_2 = \frac{118}{5}$. The maximum value of the objective function is 8326.

Activity 2

Himalayan Orchards have canned apple and bottled juice as its product with profit margin's of Rs. 2 and Rs. 1 respectively per unit. The following table indicates the labour, equipment and material to produce per unit of each product.

	Bottled Juice	Canned Apple	Total Resources
Labour (man hours)	3.0	2.0	12.0
Equipment (machine hours)	1.0	2.3	6.9
Material (unit)	1.0	1.4	4.9

Find by simplex method the product mix which will maximise the profit.

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4.4 SIMPLEX METHOD WITH SEVERAL DECISION VARIABLES

The computational procedure explained in the previous section can be readily extended to linear programming problems with more than two decision variables. This is illustrated with the help of the following example.

Example 2 The products A. B and C are produced in three machine centres X, **Y** and Z. Each product involves operation of each of the machine centres. The time required for each operation for unit amount of each product is given below. 100, 77 and 80 hours are available at machine centres X, Y and Z respectively. The profit per unit of A, B and C is **Rs.** 12, **Rs.** 3 and Rs. 1 respectively.

Find out a suitable product mix so as to maximise the profit.

Solution

The linear programming formulation of the product mix problem is as follows :

$$\text{Maximise } 12x_1 + 3x_2 + x_3$$

Subject to :

$$10x_1 + 2x_2 + x_3 \leq 100$$
$$7x_1 + 3x_2 + 2x_3 \leq 77$$
$$2x_1 + 4x_2 + x_3 \leq 80$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

We introduce **slack variables** x_4, x_5 and x_6 to make inequalities equations. Thus the problem can be stated as Maximise $12x_1 + 3x_2 + x_3$

Subject to :

$$10x_1 + 2x_2 + x_3 + x_4 = 100$$
$$7x_1 + 3x_2 + 2x_3 + x_5 = 77$$
$$2x_1 + 4x_2 + x_3 + x_6 = 80$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0$$

The first simplex Table can be obtained in a straight forward manner from the equations. We observe that the basic variables are x_4, x_5 and x_6. Therefore $C_{B1} = C_{B2} = C_{B3} = 0$.

Table 1

C_B	Basic Variables	C_j X_B	12 x_1	3 x_2	1 x_3	0 x_4	0 x_5	0 x_6
0	x_4	100	10	2	1	1	0	0
0	x_5	77	7	3	2	0	1	0
0	x_6	80	2	4	1	0	0	1
	$Z_j - C_j$		−12	−3	−1	0	0	0

1) $Z_l - C_l = -12$ is the smallest negative value. Hence x_1 should be made a basic variable in the next iteration.
2) We compute minimum of the ratios

$$\text{Min}\left[\frac{100}{10}, \frac{77}{7}, \frac{80}{2}\right] = 10$$

   The variable x_4 corresponding to which minimum occurs is made a non basic variable.
3) Table 2 is computed from Table 1 using the following rules
   a) The revised basic variables are x_{1i} x_5, x_6. Accordingly we make $C_{B1} = 12$, $C_{B2} = 0$, $C_{B3} = 0$
   b) As x_1 is the incoming basic variable we make coefficient of x_1 one by dividing each element of row 1 by 10. Thus the numerial value of the element corresponding to x_2 is $\frac{2}{10}$, corresponding to x_3 is $\frac{1}{10}$ and so on in Table 2.
   c) The incoming basic variable should appear only in the first row. So we multiply the **first row of Table 2** by 7 and subtract it from the **second row of Table I element by element.** Thus the element corresponding to x_1 in the second row of **Table 2** is zero. The element corresponding to x_2 is $3 - 7 \times \frac{2}{10} = \frac{16}{10}$.

In this way we obtain the elements of the second and the third row in Table 2. The computation of the numerical values of basic variables in Table 2 is made in a similar manner.

Table 2

C_B	Basic Variables	C_j X_B	12 x_1	3 x_2	1 x_3	0 x_4	0 x_5	0 x_6
12	x_1	10	1	1/5	1/10	1/10	0	0
0	x_5	7	0	16/10	13/10	−7/10	1	0
0	x_6	60	0	18/5	4/5	−1/5	0	1
	$Z_j - C_j$		0	−3/5	1/5	6/5	0	0

1) $Z_2 - C_2 = -\frac{3}{5}$. Hence x_2 should be made a basic variable at the next iteration.

2) We compute the minimum of the ratios

$$\text{Min}\left[\frac{10}{\frac{1}{5}}, \frac{7}{\frac{16}{10}}, \frac{60}{\frac{18}{5}}\right]$$

$$= \text{Min}\left[50, \frac{70}{16}, \frac{300}{18}\right] = \frac{70}{16}$$

3) Table 3 is computed from Table 2 following the rules indicated in a, b and c.

Table 3

C_B	Basic Variables	C_j / X_B	12 / x_1	3 / x_2	1 / x_3	0 / x_4	0 / x_5	0 / x_6
12	x_1	73/8	1	0	−1/16	3/16	−1/8	0
3	x_2	35/8	0	1	13/16	−7/16	5/8	0
0	x_6	177/4	0	0	−17/8	11/8	−9/4	1
	$Z_j - C_j$		0	0	11/16	15/16	3/8	0

As all $Z_j - C_j \geq 0$, the present solution $x_1 = 73/8$, $x_2 = 35/8$ and $x_6 = 177/4$ maximises the value of the objective function. The Maximum value of the objective function is $12 \times \frac{73}{8} + 3 \times \frac{35}{8} = \frac{981}{8}$.

Activity 3

A company manufactures three products using three types of input A, B, C in different proportions. The following Table gives the requirements of various inputs (in kg) per kilogram of the three products.

		A	B	C
Products	1	4	8	8
	2	4	6	4
	3	8	4	0

The three profit coefficients are $C_1 = 20$, $C_2 = 40$ and $C_3 = 10$. The company has 800 kg of input A, 1800 kg of input B and 500 kg of input C. Find out the product mix which will maximise the profit. What is the maximum profit?

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4.5 TWO PHASE AND M-METHOD

The simplex method illustrated in the last two sections was applied to linear programming problems with **less than or equal to** type constraints. As a result we could introduce slack variables which provided an initial basic feasible solution of the problem. Linear programming problems may also be characterised by the presence of both "less than or equal to" type or "greater than or equal to type" constraints. It may also contain some equations. Thus it is not always possible to obtain an initial basic feasible solution using slack variables.

Two methods are available to solve linear programming by simplex method in such cases. These methods will be explained with the help of numerical examples.

Two Phase Method

We illustrate the two phase method with the help of the problem presented in Activity 5 of Unit 3

Example 3

$$\text{Minimise } 12.5x_1 + 14.5x_2$$

Subject to :

$$x_1 + x_2 \geq 2000$$
$$0.4x_1 + 0.75x_2 \geq 1000$$
$$0.075x_1 + 0.1x_2 \leq 200$$
$$x_1 \geq 0, \ x_2 \geq 0$$

Solution

Although the objective function $12.5x_1 + 14.5x_2$ is to be minimised, the values of x_i and x_2 which minimised this objective function are also the values which maximise the revised objective function $-12.5x_1 - 14.5x_2$.

The second and the third constraint are multiplied by 100 and 1000 respectively for computational convenience. Thus the linear programming problem can be expressed as

$$\text{Maximise } -12.5x_1 - 14.5x_2$$

Subject to :

$$x_1 + x_2 \geq 2000$$
$$40x_1 + 75x_2 \geq 100000$$
$$75x_1 + 100x_2 \leq 200000$$
$$x_1 \geq 0, \ x_2 \geq 0$$

We convert the first two inequalities by introducing **surplus variables** x_3 and x_4 respectively. The third constraint is changed into an equation by introducing a **slack variable** x_5. Thus the linear programming problem can be expressed as

$$\text{Maximise } -12.5x_1 - 14.5x_2 = -\frac{25}{2}x_1 - \frac{29}{2}x_2$$

Subject to :

$$x_1 + x_2 - x_3 = 2000$$
$$40x_1 + 75x_2 - x_4 = 100000$$
$$75x_1 + 100x_2 + x_5 = 200000$$
$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0, \ x_5 \geq 0$$

Although **surplus variables can convert greater than or equal to type constraints into equations** they are unable to provide initial basic variables to start the simplex computation. We introduce two additional variables x_6 and x7 known as artificial variables **to facilitate the computation of an initial basic feasible solution.** The computation is carried out in two phases.

Phase I

In this phase we consider the following linear programming problem

Maximise

$$-x_6 \quad -x_7$$

Subject to :

$$x_1 + x_2 - x_3 + x_6 = 2000$$
$$40x_1 + 75x_2 - x_4 + x_7 = 100000$$
$$75x_1 + 100x_2 + x_5 = 200000$$
$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0, \ x_5 \geq 0, \ x_6 \geq 0, \ x_7 \geq$$

An initial basic feasible solution of the problem is given by $x_6 = 2000$, x_7=100000,

x_5 200000. As the minimum value of the Phase I objective function is zero at the end of the Phase I computation both x, and x_7 become zero.

Phase II

The basic feasible solution at the end of Phase I computation is used as the initial basic feasible of the problem. The original objective function is introduced in Phase II computation and the usual simplex procedure is used to solve the problem.

Phase I Computation

Table 1

C_B	Basic Variables	C_j / X_B	0 / x_1	0 / x_2	0 / x_3	0 / x_4	0 / x_5	−1 / x_6	−1 / x_7
−1	x_6	2000	1	1	−1	0	0	1	0
−1	x_7	100000	40	75	0	−1	0	0	1
0	x_5	200000	75	100	0	0	1	0	0
		$Z_j - C_j$	−41	−76	1	1	0	0	0

x2 becomes a basic variable and x_7 becomes a non basic variable in the l.ext iteration. It is no longer considered for re-entry.

Table 2

C_B	Basic Variables	C_j / X_B	0 / x_1	0 / x_2	0 / x_3	0 / x_4	0 / x_5	−1 / x_6
−1	x_6	$\frac{2000}{3}$	$\frac{7}{15}$	0	−1	$\frac{1}{75}$	0	1
0	x_2	$\frac{4000}{3}$	$\frac{8}{15}$	1	0	$-\frac{1}{75}$	0	0
0	x_5	$\frac{200000}{3}$	$\frac{65}{3}$	0	0	$\frac{4}{3}$	1	0
		$Z_j - C_j$	$-\frac{1}{15}$	0	1	$-\frac{7}{75}$	0	0

x_1 becomes a basic variable and x_6 becomes a non basic variable in the next iteration. It is no longer considered for re-entry.

Table 3

C_B	Basic Variables	C_j / X_B	0 / x_1	0 / x_2	0 / x_3	0 / x_4	0 / x_5
0	x_1	$\frac{10000}{7}$	1	0	$-\frac{15}{7}$	$\frac{1}{35}$	0
0	x_2	$\frac{4000}{7}$	0	1	$\frac{8}{7}$	$-\frac{1}{35}$	0
0	x_5	$\frac{250000}{7}$	0	0	$\frac{325}{7}$	$\frac{16}{21}$	1
		$Z_j - C_j$	0	0	0	0	0

The Phase I computation is complete at this stage. Both the artificial variables have been removed from the basis. We have also found a basic feasible solution of the problem, namely, $x_1 = \frac{10000}{7}$, $x_2 = \frac{4000}{2}$, $x_5 = \frac{250000}{7}$. In Phase II computation we use the actual objective function of the problem.

Phase 11 Computation

Table 1

C_B	Basic Variables	C_j / X_B	$-\frac{25}{2}$ / x_1	$-\frac{29}{2}$ / x_2	0 / x_3	0 / x_4	0 / x_5
−25/2	x_1	$\frac{10000}{7}$	1	0	$-\frac{15}{7}$	$\frac{1}{35}$	0
−29/2	x_2	$\frac{4000}{7}$	0	1	$\frac{8}{7}$	$-\frac{1}{35}$	0
0	x_5	$\frac{250000}{7}$	0	0	$\frac{325}{7}$	$\frac{5}{7}$	1
		$Z_j - C_j$	0	0	$\frac{143}{14}$	$\frac{2}{35}$	0

As all $Z_j - C_j \geq 0$ the current solution maximises the **revised** objective function. Hence the solution of the problem is given by $x_1 = \frac{10000}{7} = 1428\frac{4}{7}$, $x_2 = \frac{4000}{7} = 571\frac{3}{7}$. The minimum value of the objective function is $26142\frac{6}{7}$. The problem has been solved earlier by graphical method in Unit 3.

M-method

The M-method also uses artificial variables for locating an initial basic feasible solution. We illustrate this method with the help of the previous example.

$$\text{Maximise } -\frac{25}{2}x_1 - \frac{29}{2}x_2$$

Subject to :

$$x_1 + x_2 - x_3 = 2000$$
$$40x_1 + 75x_2 - x_4 = 100000$$
$$75x_1 + 100x_2 + x_5 = 200000$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

We introduce artificial variables $x_6 \geq 0$, $x_7 \geq 0$ to the first and the second constraint respectively. The objective function is revised using **a large positive number M.** Thus instead of the original linear programming problem the following linear programming problem is considered

$$\text{Maximise } -25/2x_1 - 29/2x_2 - M(x_6 + x_7)$$

Subject to :

$$x_1 + x_2 - x_3 + x_6 = 2000$$
$$40x_1 + 75x_2 - x_4 + x_7 = 100000$$
$$75x_1 + 100x_2 + x_5 = 200000$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0,\ x_7 \geq 0.$$

The coefficients of the artificial variables in the objective function are large negative numbers. As the objective function is to 1 maximised in the **optimum or optimal solution** (where the objective function is maximised) the artificial variables will be zero. The basic variables of the optimal solution are therefore variables other than artificial variables and hence is a basic solution of the original problem. The successive simplex Tables are given below :

Table 1

C_B	Basic	C_j	$-\frac{25}{2}$	$-\frac{29}{2}$	0	0	0	$-M$	$-M$
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-M$	x_6	2000	1	1	-1	0	0	1	0
$-M$	x_7	100000	40	75	0	-1	0	0	1
0	x_5	200000	75	100	0	0	1	0	0
	$Z_j - C_j$		$-41M + \frac{25}{2}$	$-76M + \frac{29}{2}$	M	M	0	0	0

As M is a large positive number, the **coefficient of** M **in the Z_j C_j row would decide the incoming basic variable.** As -76M < -41M, x_2 becomes a basic variab16 in the next iteration replacing x_7. The variable x_7 being an artificial variable it is not considered for re-entry as a basic variable.

Table 2

C_B	Basic Variables	C_j X_B	$-25/2$ x_1	$-29/2$ x_2	0 x_3	0 x_4	0 x_5	$-M$ x_6
$-M$	x_6	2000/3	7/15	0	-1	1/75	0	1
$-29/2$	x_2	4000/3	8/15	1	0	$-1/75$	0	0
0	x_5	200000/3	65/3	0	0	4/3	1	0
		$Z_j - C_j$	$-7/15M + 143/30$	0	M	$-M/75 + 29/150$	0	0

x_1 becomes a basic variable replacing x_6. The variable x_6 being an artificial variable is not considered for re-entry as a basic variable.

Table 3

C_B	Basic Variables	C_j / X_B	$-25/2$ / x_1	$-29/2$ / x_2	0 / x_3	0 / x_4	0 / x_5
$-25/2$	x_1	10000/7	1	0	$-15/7$	1/35	0
$-29/2$	x_2	4000/7	0	1	8/7	$-1/35$	0
0	x_5	250000/7	0	0	325/7	16/21	1
		$Z_j - C_j$	0	0	143/14	2/35	0

Hence the optimum solution of the problem is $x_1 = 10000/7 = 1428\frac{4}{7}$, $x_2 = 4000/7 = 571\frac{3}{7}$ with the minimum value of the objective function being $26142\frac{6}{7}$.

Activity 4

Solve the following linear programming problem by **Two-Phase Method and. M-method** using artificial variables corresponding to second and third constraints.

Maximise $12x_1 + 15x_2 + 9x_3$

Subject to :

$$8x_1 + 16x_2 + 12x_3 \leq 250$$
$$4x_1 + 8x_2 + 10x_3 \geq 80$$
$$7x_1 + 9x_2 + 8x_3 = 105$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

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4.6 MULTIPLE, UNBOUNDED SOLUTIONS AND INFEASIBLE PROBLEMS

The simplex method can identify multiple solutions of a linear programming problem. If a problem possesses an unbounded solution it is also located in course of simplex computation. If a linear programming problem is infeasible it is revealed by simplex computation. We illustrate these applications of simplex method with the help of a number of examples.

Example 4

We consider the linear programming problem formulated in Unit 3, Section 7.

Maximise $2000x_1 + 3000x_2$

Subject to :

$$6x_1 + 9x_2 \leq 100$$
$$2x_1 + x_2 \leq 20$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

Solution

After introducing slack variables x_3 i 0, x_4 0 the inequalities can be converted into equations as follows

$$6x_1 + 9x_2 + x_3 = 100$$
$$2x_1 + x_2 + x_4 = 20$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

The successive tables of simplex computation are shown below :

Table 1

C_B	Basic Variables	C_j X_B	2000 x_1	3000 x_2	0 x_3	0 x_4
0	x_3	100	6	9	1	0
0	x_4	20	2	1	0	1
	$Z_j - C_j$		−2000	−3000	0	0

Table 2

	Basic Variables	C_j X_B	2000 x_1	3000 x_2	0 x_3	0 x_4
00	x_2	100/9	2/3	1	1/9	0
	x_4	80/9	4/3	0	−1/9	1
	$Z_j - C_j$		0	0	3000/9	0

Since $Z_j - C_j \geq 0$ for all the variables, $x_1 = 0$, $x_2 = 100/9$ is an optimum solution of the problem. The maximum value of the objective function is 100000/3. However, the Z_3 - C_3 value corresponding to the **non basic variable x_1 is also zero.** This indicates that there is more than one optimum solution of the problem. In order to compute .the value 9f the alternative optimum solution we Introduce x_1 as a basic variable replacing x4. The subsequent computation is presented in the next Table.

C_B	Basic Variables	C_j X_B	2000 x_1	3000 x_2	0 x_3	0 x_4
3000	x_2	20/3	0	1	1/6	1/2
2000	x_1	20/3	1	0	−1/12	3/4
	$Z_j - C_j$		0	0	1000/3	3000

Thus $x_1 = 20/3$, $x_2 = 20/3$ also maximise the objective function. The maximum value as in the previous solution is 100000/3.

Example 5

Consider the linear programming problem

$$\text{Maximise } 5x_1 + 4x_2$$

Subject to :

$$x_1 \leq 7$$
$$x_1 - x_2 \leq 8$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Solution

After introducing slack variables x_3 0, x_4 0 the corresponding equations are

$$x_1 + x_3 = 7$$
$$x_1 - x_2 + x_4 = 8$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0.$$

The successive simplex iterations are shown below :

Table 1

C_B	Basic Variables	C_j / X_B	5 / x_1	4 / x_2	0 / x_3	0 / x_4
0	x_3	7	1	0	1	0
0	x_4	8	1	−1	0	1
	$Z_j - C_j$		−5	−4	0	0

Table 2

C_B	Basic Variables	C_j / X_B	5 / x_1	4 / x_2	0 / x_3	0 / x_4
5	x_1	7	1	0	1	0
0	x_4	1	0	−1	−1	1
	$Z_j - C_j$		0	−4	5	0

$Z_2 - C_2 < 0$ indicates x_2 should be introduced as a basic variable in the next iteration.

However, both $y_{12} \leq 0, y_{22} \leq 0$. Thus it is not possible to proceed with the simplex computation any further as you cannot decide which variable will be non basic at the next iteration. This is the criterion for **unbounded solution.**

If in the course of simplex computation $Z_j - C_j < 0$ but $y_{ij} \leq 0$ for all i then the problem has no finite solution.

Intuitively, you may observe that the variable x_2 in reality is unconstrained and can be increased arbitrarily. This is why the solution is unbounded.

Example 6

We consider the linear programming problem formulated in Unit 3, Section 6.

Minimise $200x_1 + 300x_2$

Subject to :

$$2x_1 + 3x_2 \geq 1200$$
$$x_1 + x_2 \leq 400$$
$$2x_1 + 3/2x_2 \geq 900$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Solution

After converting the minimisation problem into a maximisation problem and introducing slack, surplus, artificial variables the problem can be presented as

Maximise $-200x_1 - 300x_2$

Subject to :

$$2x_1 + 3x_2 - x_3 + x_6 = 1200$$
$$x_1 + x_2 + x_4 = 400$$
$$2x_1 + 3/2x_2 - x_5 + x_7 = 900$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0,\ x_7 \geq 0.$$

The variables x_6 and x_7 are artificial variables. We use two phase method to solve this problem. In phase I, we use the objective function :

Maximise -x6 - x7

along with the constraints given above. The successive simplex computations are given below

Table 1

C_B	Basic Variables	C_j / X_B	0 / x_1	0 / x_2	0 / x_3	0 / x_4	0 / x_5	−1 / x_6	−1 / x_7
−1	x_6	1200	2	3	−1	0	0	1	0
0	x_4	400	1	1	0	1	0	0	0
−1	x_7	900	2	3/2	0	0	−1	0	1
	$Z_j - C_j$		−4	−9/2	1	0	1	0	0

Table 2

C_B	Basic Variables	C_j X_B	0 x_1	0 x_2	0 x_3	0 x_4	0 x_5	−1 x_7
0	x_2	400	2/3	1	−1/3	0	0	0
0	x_4	0	1/3	0	1/3	1	0	0
−1	x_7	300	1	0	1/2	0	−1	1
	$Z_j - C_j$		−1	0	−1/2	0	1	0

Table 3

C_B	Basic Variables	C_j X_B	0 x_1	0 x_2	0 x_3	0 x_4	0 x_5	−1 x_6
0	x_2	400	0	1	−1	−2	0	0
0	x_1	0	1	0	1	3	0	0
−1	x_7	300	0	0	−1/2	−3	−1	1
	$Z_j - C_j$		0	0	1/2	3	1	0

Thus $Z_j - C_j \geq 0$ for all the variables but the artificial variable x_7 is still a basic variable. This indicates that the problem has no feasible solution.

If in course of simplex computation by two phase method one or more artificial variables remain basic variables at the end of Phase I computation, the problem has no feasible solution.

Activity 5

Solve the linear programming problem by simplex Method and give your comments.

Maximise $x_1 + x_2$

Subject to :

$$-2x_1 + x_2 \leq 1$$
$$x_1 \leq 2$$
$$x_1 + x_2 \leq 3$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

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Activity- 6

Solve the following linear programming problem by simplex method and give your comments.

Maximise $3x_1 + 2x_2$

Subject to :

$$x_1 - x_2 \leq 1$$
$$x_1 + x_2 \geq 3$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

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Activity 7

Solve the following linear programming problem by two phase method and give your

comments.

Maximise $x_1 + x_2$

Subject to :

$$x_1 + x_2 \leq 1$$
$$-3x_1 + x_2 \geq 3$$
$$x_1 \geq 0, \; x_2 \geq 0$$

4.7 SENSITIVITY ANALYSIS

After a linear programming problem has been solved, it is useful to study the effect of changes in the parameters of the problem on the current optimal solution. Some typical situations are the impact of the changes in the profit or cost in the objective function in the current solution or an increase or decrease in the resource level in the present composition of the product mix. Such an investigation can be carried out from the final simplex Table and is known as **sensitivity analysis or post optimality analysis.** We shall illustrate this method with the help of an example

Example 7

We consider the linear programming problem introduced in Section 4.4

Maximise $12x_1 + 3x_2 + x3$

Subject to :

$$10x_1 + 2x_2 + x_3 \leq 100$$
$$7x_1 + 3x_2 + 2x_3 \leq 77$$
$$2x_1 + 4x_2 + x_3 \leq 80$$
$$x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0.$$

The final simplex Table presenting the optimum solution '(x_4, x_5, x_6 being the slack variables) is presented below :

Table 1

C_B	Basic Variables	C_j / X_B	12 / x_1	3 / x_2	1 / x_3	0 / x_4	0 / x_5	0 / x_6
12	x_1	73/8	1	0	1/16	3/16	–1/8	0
3	x_2	35/8	0	1	13/16	–7/16	5/8	0
0	x_6	177/4	0	0	–17/8	11/8	–9/4	1
	$Z_j - C_j$		0	0	11/16	15/16	3/8	0

Change in the Profit coefficient

i) Non Basic Variable

The variable x_3 non basic. If its profit level is increased to C_3 then the solution remained unchanged so long as

$$Z_3 - \bar{C}_3 \geq 0$$

i.e. $$Z_3 - C_3 + C_3 - \bar{C}_3 \geq 0$$

i.e. $$\bar{C}_3 \leq C_3 + Z_3 - C_3 = 1\frac{11}{16}$$

If $C_3 > 1\ 16$ then **the present solution is no longer** optimum; A new round of simplex computation is to be performed in which x_3 becomes a basic variable;

However, if $C_3 < 1\ 16$, the present solution remains optimum.

ii) Basic Variable

In this case, the changes can be both positive and negative, as in either case the current solution may become non-optimal. Let us consider the basic variable x_1. Let us define by $\Delta 1 = \bar{C}_1 - 12$ as the change (both positive and negative) in the profit coefficient 12. **We divide each Z_j-C_j value of the non basic variable by the corresponding coefficient in the x_1 row** which is denote by $\frac{Zj\text{-}Cj}{y_{1j}}$. Then the basic variables remain unchanged so long, as (Mustafi, 1988)

$$-\text{Minimum}\left[\frac{Z_j - C_j}{y_{1j}};\ y_{1j} > 0\right] \leq \Delta_1 \leq \underset{j}{\text{Minimum}}\left[\frac{Z_j - C_j}{-y_{1j}};\ y_{ij} < 0\right]$$

Referring to the final simplex Table, we observe that corresponding to the non basic variables x_3 and x_5, $y_{13} = -\frac{1}{16}$, $y_{15} = -\frac{1}{8}$.

Hence,

$$\text{Minimum}\left[\frac{Z_j - C_j}{-y_{1j}};\ y_{1j} < 0\right]$$

$$= \underset{j}{\text{Minimum}}\left[\frac{11/16}{1/16}, \frac{3/8}{1/8}\right]$$

$$= \text{Minimum}\ (11, 3) = 3.$$

Corresponding to the non basic variables x_4, $y_{14} > 0$.

Hence

$$\underset{j}{\text{Minimum}}\left[\frac{Z_j - C_j}{y_{1j}};\ y_{1j} > 0\right] = \frac{15/16}{3/16} = 5$$

Hence

$$-5 \leq \bar{C}_1 - 12 \leq 3 \quad \text{i.e.} \quad 7 \leq \bar{C}_1 \leq 15.$$

Thus the optimal solution is insensitive so long as the changed profit coefficient $\bar{C}_1$ is between Rs. 7 and Rs. 15 although the present profit coefficient is Rs. 12. Of course, the value of the objective function has to be revised after introducing the change value $\bar{C}_1$.

Activity 8

In the example considered in Section 4.7 find the range of the profit coefficient of x_2 within which the present solution remains optimum.

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4.8 DUAL LINEAR PROGRAMMING PROBLEM

Every linear programming problem is associated with another linear programming problem known as its **dual problem.** The original problem in this context is known as the **primal** problem. The formulation of the dual linear programming problem (also sometimes referred to as the concept of **duality)** is substantially helpful to our understanding of linear programming. The variables of the dual linear programming problem also known as **dual variables have** important economic interpretations which can be_ used by a decision maker for planning his resources. Under certain circumstances the dual problem is easier to solve than the primal problem. The solution of the dual problem leads to the solution of the primal problem and thus efficient computational techniques can be developed through the concept of duality. Finally in the problems of competitive strategy solution of both the primal and the dual problem is necessary to understand the problem fully.

We introduce the concept of duality with the help of the product mix problem introduced in Section 4.4.

Example 8

Three products A, B, C are produced in three machine centres x, y, z. Each product involves operation of each of the machine centres. The time required for each operation on various products is indicated in the following Table. 100, 77 and 80 hours are only available at machine centres x, y and z respectively. The profit per unit of A, B and C is Rs. 12, Rs. 3 and Re 1 respectively.

Table showing the data of the Product mix problem

Products	Machine Centres X	Y	Z	Profit per unit
A	10	7	2	Rs. 12
B	2	3	4	Rs. 3
C	1	2	1	Re 1
Available hours	100	77	80	

Solution

The linear programming formulation or the **primal problem** is given by

$$\text{Maximise } 12x_1 + 3x_2 + x_3$$

Subject to .

$$10x_1 + 2x_2 + x_3 \leq 100$$
$$7x_1 + 3x_2 + 2x_3 \leq 77$$
$$2x_1 + 4x_2 + x_3 \leq 80$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

After introducing the slack variables $x_4 >= 0$, $x_5 >= 0$, $x6>=0$ and using simplex method we obtain the optimum solution as $x_1 = 73/8$, $x_2 = 35/8$ and the maximum value of the objective function as 981/8. The final Simplex Table is given below :

Final Simplex Table

C_B	Basic Variables	C_j / X_B	12 / x_1	3 / x_2	1 / x_3	0 / x_4	0 / x_5	0 / x_6
12	x_1	73/8	1	0	−1/16	3/16	−1/8	0
3	x_2	35/8	0	1	13/16	−7/16	5/8	0
0	x_6	177/4	0	0	−17/8	11/8	−9/4	1
		$Z_j - C_j$	0	0	11/16	15/16	3/8	0

Suppose a prospective investor is planning to purchase the resources x, y, z. What offer is he going to make to manufacturer? Let us assume that W_1, W_2 and W_3 are the offers made per hour of machine time x, y and z respectively. Then these prices W_1, W_2, W_3 must satisfy the following conditions.

a) $W_1 \geq 0, \quad W_2 \geq 0, \quad W_3 \geq 0.$

b) Assuming that the prospective investor is behaving in a rational manner, he would try to bargain as much as possible and hence the total amount payable to the manufacturer would be as tittle as possible. This leads to the condition :

   Minimise $100W_1 + 77W_2 + 80W_3$

c) The total amount offered by the prospective investor to the three resources required to produce one unit of each product must be at least as high as the profit earned by the manufacturer per unit. Since these resources enable the manufacturer to earn the specified profit corresponding to the product he would not like to sell it for anything less assuming he is behaving rationally. This leads to inequalities.

$$10w_1 + 7w_2 + 2w_3 \geq 12$$
$$2w_1 + 3w_2 + 4w_3 \geq 3$$
$$w_1 + 2w_2 + w_3 \geq 1$$

We have, thus, a linear programming problem to ascertain the values of the variables w_1, w_2, w3. The variables are known as **dual variables.** The primal problem presented in this example (i) considers maximisation of the objective function (ii) has less than or equal to type constraints and (iii) has non-negativity constraints on the variables. Such a problem is known as a **primal problem in the standard form.**

Formulation of a dual problem

If the primal problem is in the standard form, the, dual problem can be formulated using the following 'rules.

1) The number of constraints in the primal problem is equal to the number of dual variables. The number of constraints in the dual problem is equal to the number of variables in the primal problem. The primal problem is a maximisation problem, the dual problem is a minimisation problem.
2) The profit coefficients of the primal problem appear on the right hand side of the constraints of the dual problem.
3) The primal problem has less than or equal to type constraints while the dual problem has greater than or equal to type constraints.
4) The coefficients of the constraints of the primal problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and. vice versa.

It is easy to verify these rules with respect to the example discussed before.

Properties of the dual problem (Mustafi, 1988)

1) If the primal problem is in the standard form the solution of the dual problem can be obtained from the Z_i - C_l values of the slack variables in the final simplex Table.

Example

In the example discussed previously the variables x_4, x_5, x_6 are slack variables. Hence the solution of the dual problem is $w_1 = z_4\ c_4 = 15/16$, $w_2 = 3/8$, $w_3 =$ O.

2) The maximum value of the objective function of the primal problem is the minimum value of the objective function of the dual problem.

   Example

   The maximum value of the objective of the primal problem is 981/8. The minimum value of the objective function of the dual problem is

$$100 \times \frac{15}{16} + 77 \times \frac{3}{8} + 80 \times 0$$
$$= \frac{981}{8}$$

   The result has an important practical implication, If the problem is analysed by both the manufacturer and the investor then neither of the two can outmanoeuver the other.

   Shadow price

   The **shadow price** of a resource is the unit price that is equal to the increase in profit to be realised by one additional unit0 of the resource.

Example

The maximum value of the objective function can be expressed as

$$100 \times \frac{15}{16} + 77 \times \frac{3}{8} + 80 \times 0$$

If the first type of resource is increased by one unit the maximum profit will increase by 15/16 which is the value of the first dual variable in the optimum solution.

Thus the dual variable is also referred to as the **shadow price** or **imputed price** of a resource. This is the **highest price** the manufacturer would be willing to pay for the resource. The shadow price of the third resource is zero as there is already an unutilised amount; profit is not increased by more of it until the current supply is totally exhausted.

3) Suppose the number of constraints and variables in the primal problem is m and n respectively. The number of constraints and variables in the dual problem is, therefore, n and m respectively. Suppose the slack variables in the primal are denoted by $y_1, y_2, \ldots, y_n$ and the surplus variables in the dual problem are denoted by $z_1, z_2, \ldots, z_n$

a) **In the optimum solution**

if $x_i > 0, \quad z_i = 0, \quad i = 1, 2, \ldots, n.$

if $z_i > 0, \quad x_i = 0, \quad i = 1, 2, \ldots, n.$

if $w_i > 0, \quad y_i = 0, \quad i = 1, 2, \ldots, n.$

if $y_i > 0, \quad w_i = 0, \quad i = 1, 2, \ldots, n.$

b) If a solution of the primal and the corresponding dual problem satisfy the above conditions then it must be **an optimum solution.**

This result is commonly referred to as the **complementary slackness condition**

Referring to the final simplex Table of the problem discussed before we observe m = 3, n = 3. In the optimum solution

$x_1 = 73/8 \qquad x_2 = 35/8 \qquad x_3 = 0$

$z_1 = 0 \qquad z_2 = 0 \qquad z_3 = 0$

$y_1 = 0 \qquad y_2 = 0 \qquad y_3 = 177/4$

$w_1 = 15/16 \qquad w_2 = 3/8 \qquad w_3 = 0$

Thus, the complementary slackness condition is satisfied.

4) If the primal problem is in the **non standard** form, the stricture he dual problem remains unchanged. However, if a constraint is **greater than equal to type, the corresponding dual variable is negative or zero.** If a constraint in the primal problem is **equal to type,** the corresponding dual variable is **unrestricted in sign** (may be positive or negative).

Example 9

Consider the primal linear programming problem

Maximise $12x_1 + 15x_2 + 9x_3$

Subject to :

$8x_1 + 16x_2 + 12x_3 \leq 250$

$4x_1 + 8x_2 + 10x_3 \geq 80$

$7x_1 + 9x_2 + 8x_3 = 105$

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$

The corresponding dual problem is

Minimise $250w_1 + 80w_2 + 105w_3$

Subject to :

$8w_1 + 4w_2 + 7w_3 \geq 12$

$$16w_1 + 8w_2 + 9w_3 \geq 15$$
$$12w_1 + 10w_2 + 8w_3 \geq 9$$
$$w_1 \geq 0, \quad w_2 \leq 0, \quad w_3 \text{ unrestricted in sign.}$$

Activity 9

A garment manufacturer has a production line making two styles of shirts. Style I requires 200 grams of cotton thread, 300 grams of dacron thread, and 300 grams of linen thread. Style II requires 200 grams of cotton thread, 200 grams of dacron thread and 100 grams of linen thread. The manufacturer makes a net profit of Rs. 19.50 on Style I, Rs. 15.90 on Style II. He has in hand an inventory of 24 kg of cotton thread, 26 kg of dacron thread and 22 kg of linen thread. His immediate problem is to determine, a production schedule, given the current inventory to make a maximum profit. Then he would like to know at what price it would be profitable to buy thread

Solve the problem and explain how the concept of duality can be helpful to find out the right price for various kinds of thread.

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4.9 SUMMARY

The **simplex method** is the appropriate method for solving a linear programming problem with more than two decision variables. For less than or equal to type **constraints slack variables** are introduced to make inequalities equations. A particular type of solution known as a **basic feasible solution** is important for simplex computation." Every basic feasible solution is an extreme point of the convex set of feasible solutions and vice versa. A basic feasible solution of a system with m equations and n variables has m non negative variables known as **basic variables** and n-m variables with value zero known as **non-basic variables,** We can always find a basic feasible solution with the help of the slack variables. **The objective function is maximised or minimised at one of the basic feasible solutions.** Starting with the initial basic feasible solution obtained from the slack variables the simplex method improves the value of the objective function step by step by bringing in a new basic variable and making one of the present basic variables non basic. The selection of the new basic variable and the omission of a current basic variable are performed following certain rules so that the revised basic feasible solution improves the value of the objective function. The iterative procedure stops when it is no longer possible to obtain a better value of the objective function than the present one. The existing basic feasible solution is the **optimum** solution of the problem which maximises or minimises the objective function as the case may be.

When one or more of the constraints are greater than or equal to type **surplus variables** are introduced to make inequalities equations. Surplus variables, however. cannot be used to obtain an initial basic feasible solution. If some of the constraints are greater than or equal to type or equations **artificial variables** are used to initiate the simplex computation Two methods, namely, **Two Phase Method** and **M-method** are available to solve linear programming problems in these cases. The simplex method can identify **multiple** or **unbounded solutions and infeasible problems.**

The simplex method also provides a mean for carrying out **sensitivity** or **post optimality** analysis of the problem. It is possible to study the effect of change in profit contribution for a particular product without solving the problem all over again. The effect of change in various resource levels can also be ascertained by making a few additional calculations.

Every linear programming problem has an accompanying linear programming problem known as a **dual problem.** The variables of the dual problem are known as **dual variables.** The dual variables have an economic interpretation which can be used by management: for planning its resources. The solution of the dual problem can be obtained from the simplex computation of the original problem. The solution has a number of important properties which can also. be helpful for computational purposes.

4.10 KEY WORDS

A Slack Variable corresponding to a less than or equal to type constraint is a non negative variable introduced to convert the constraint into an equation.

A Basic Solution of a system of m equations and n variables (m < n) is a solution where at least n-m variables are zero.

A **Basic Feasible Solution** of a system of m equations and n variables (m < n) is a solution where m variables are non negative and n-m variables are zero.

A Basic Variable of a basic feasible solution has a non negative value.

A Non Basic Variable of a basic feasible solution has a value equal to zero.

A Surplus Variable corresponding to a greater than or equal to type constraint is a non negative variable introduced to convert the constraint into an equation.

An Artificial Variable is a non negative variable introduced to facilitate the computation of an initial basic feasible solution.

The Optimum Solution of a linear programming problem is the solution where the objective function is maximised or minimised.

The Sensitivity Analysis of a linear programming problem is a study of the effect of changes of the profit or resource level on the solution.

The Dual Problem corresponding' to a linear programming problem is another linear programming problem formulated from the parameters of the original problem.

The Primal Problem is the original linear programming problem.

The Dual Variables are the variables of the dual linear programming problem.

The Shadow Price of a resource is the change in the optimum value of the objective function per unit increase of the resource.

4.11 SELF-ASSESSMENT EXERCISES

1) A manufacturer has production facilities for assembling two different types of television sets. These facilities can be used to assemble both black and white and coloured sets. At the present time the firm is producing only one model of each type of set. The black and white set contributes Rs. 150 towards profit while a coloured set contributes Rs. 450 towards profit. The number of coloured television sets manufactured everyday cannot exceed 50 as the number of coloured picture tubes available everyday is 50. Each black and white set requires 6 man-hours of chassis assembly time, whereas each coloured set requires 18 man hours. The daily available man hours for the chassis assembly line is 1800. A black and white set must spend one man hour on. the set assembly line whereas a coloured set must spend 1.6 man hours on the set assembly line. The daily available man hours on this line is 240. A black and white television set requires 0.5 man hours of testing- and final inspection whereas a coloured set requires 2 man hours. The total available man hours per day for testing and inspection is 162.

   Formulate and solve the problem using **simplex method** so that the profit is maximised.

2) A small scale unit is in a position to manufacture three products A, B and C. Raw material required per piece of product A, B and C is 2 kg, 1 kg, and 2 kg respectively while the total daily availability is 50 kg. The raw material is processed on a machine by the labour force and on a day the availability of machine hours is 30 while the availability of labour hour is 26. The time required per unit production of the three products is given below

Product	Machine Hour	Labour Hour
A	$\frac{1}{2}$	1
B	3	2
C	1	1

The net per unit profit from **products** A, **B,** C **respectively** are Rs. 25, Rs. 30 and Rs. 40. Find the linear **programming** formulation of the problem. Solve the problem by **simplex method** to obtain the maximum profit per day.

3) A small **electronics dealer buys** various **components** to **assemble** them into transistors, **tape recorders** and small stereo sets. In **a week** the dealer has time to assemble at the most 500 units of any one or, the combined items. Transistors and tape **recorders have a weekly combined order** of at least 150 units. Transistors being very **popular,** the **number** of these units **assembled** must exceed the number of ' tape recorders and stereos combined exactly by 100 units. The contribution towards profit **made by each transistor, tape** recorder **and** stereo is Rs.75, Rs. 125 and Rs. 150 **respectively.** How **many** units of **these** musical items be assembled each week to maximise the total profit?

4) Solve the following linear programming problem and give your comments

Maximise $6x_1 + 2x_2 + 4x_3$

Subject to :

$$2x_1 + 3x_2 + x_3 \leq 28$$
$$3x_1 + x_2 + 2x_3 \leq 24$$
$$x_1 + 2x_2 + 3x_3 \leq 35$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

5) Solve the following problem by simplex method and give your comments.

Maximise $3x_1 + 2x_2$

Subject to :

$$x_1 - x_2 \leq 1$$
$$3x_1 - 2x_2 \leq 6$$
$$x_1 \geq 0,\ x_2 \geq 0$$

6) Solve the linear programming problem and give your comments

Maximise $4x_1 + 5x_2$

Subject to :

$$2x_1 + 3x_2 \geq 12$$
$$x_1 + x_2 \leq 4$$
$$4x_1 + 3x_2 \geq 18$$
$$x_1 \geq 0,\ x_2 \geq 0$$

7) In Activity 3 find the range within which the profit coefficients of the three decision variables can vary without changing the optimum solution.

8) Formulate the dual linear programming problem of Exercise 2 and find the optimum values of the dual variables.

9) Write down the primal problem corresponding to the dual problem given below and hence find its solution

Minimise $3w_1 + 4w_2$

Subject to :

$$3w_1 + 4w_2 \geq 24$$
$$2w_1 + w_2 \geq 10$$
$$5w_1 + 3w_2 \geq 29$$
$$w_1 \geq 0,\ w_2 \geq 0$$

4.12 ANSWERS

Activity 1

i) slack, less than or equal to type

ii) ^nC_m

iii) $n - m$

iv) non negative

v) ^nC_m

vi) extreme, basic feasible

vii) basic feasible

Activity 2

Canned apple $\frac{20}{9}$. Bottled juice $\frac{161}{90}$.

Maximum profit $\frac{187}{30}$.

Activity 3

$x_1 = 0$, $x_2 = 125$, $x_3 = 75/2$. Maximum Profit 5375.

Activity 4

$x_1 = 6$, $x_2 = 7$, $x_3 = 0$. Maximum 177.

Activity 5

$x_1 = 2$, $x_2 = 1$ or $x_1 = 2/3$, $x_2 = 7/3$.
Maximum value of the objective function 3.

Activity 6

Unbounded solution

Activity 7

Infeasible problem

Activity 8

$12/5 \leq \bar{C}_2 \leq 36/7$

Activity 9

Style I 20, Style II 100 Maximum Profit Rs. 1980

Price per kg. of cotton Rs. 43.50

Price per kg. of dacron Rs. 36.00

Price per kg. of linen = 0

Self-assessment Exercises

1) x_1 : No. of Black and White TV

x_2 : No. of Coloured Television Set

Maximise $150x_1 + 450x_2$

Subject to :

$x_2 \leq 50$

$x_1 + 1.6x_2 \leq 240$

$0.5x_1 + 2x_2 \leq 162$

$x_1 \geq 0$, $x_2 \geq 0$

$x_1 = 184$, $x_2 = 35$

Maximum Profit = Rs. 433.50

2) Product B 2/3

   Product C 74/3

   Maximum Profit Rs. $1006\frac{2}{3}$

3) No. of transistors = 300

   No. of tape recorders = 0

   No. of stereo sets = 200

   Maximum value of the objective function = Rs. 52500

4) $x_1 = 8$ or $x_1 = 44/7$, $x_2 = 36/7$ or $x_1 = 20/7$, $x_3 = 54/7$

   Maximum value of the objective function is 48.

5) Unbounded solution

6) Infeasible Problem

7) $\bar{C}_1 \leq 75$ $\bar{C}_2 \geq 12.5$ $0 \leq \bar{C}_3 \leq 80$

8) Minimise $50w_1 + 30w_2 + 26w_3$

   Subject to :

$$2w_1 + \frac{1}{2}w_2 + w_3 \geq 25$$

$$w_1 + 3w_2 + 2w_3 \geq 30$$

$$2w_1 + w_2 + w_3 \geq 40$$

$$w_1 = 50/3, \quad w_1 = 0, \quad w_3 = 20/3$$

9) Primal Problem

   Maximum $24x_1 + 10x_2 + 29x_3$

   Subject to :

$$3x_1 + 2x_2 + 5x_3 \leq 3$$

$$4x_1 + x_2 + 3x_3 \leq 4$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

$$w_1 = 4, \quad w_2 = 3$$

   Minimum value of the objective function 24.

4.13 FURTHER READINGS

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