Multivariate Calculus

Book : M. J. Strauss, G.L. Bradley and K.J. Smith, Calculus (3rd Edition), Dorling Kindersley (India) Pvt Ltd.

11.1 Functions of several variables

$$f : (x,y) \longrightarrow f(x,y)$$

D (domain) Range

$$z = f(x,y)$$

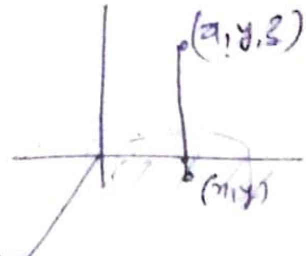
Graph of $z = f(x,y)$ is a surface in $\mathbb{R}^3$.

There are 2 independent variables x and y and one dependent variable z.

In a fn of one variable $y = f(x)$, there is one independent variable x which is plotted on axis and one dependent variable y which is plotted on y axis. So graph is 2D

For fn of 2 variable $z = f(x,y)$, domain is subset of $\mathbb{R}^2$, i.e every pt $(x,y) \in D$ is 2-dim i.e needs xy plane to plot.

and graph will be 3D.



graph of $f(x,y) = x^2+y^2$ or $z = x^2+y^2$ — 3D

$g(x,y,z) = \dfrac{x^2+y^2}{z-3}$ or $w = \dfrac{x^2+y^2}{z-3}$ is 4D (domain is 3D)

$h(x) = x+1$ or $y = x+1$ is 2D. (domain is 1D)

$f(x,y,z) = \sqrt{x^2+2y^2+3z^2}$: 3 independent variables
domain is 3D
graph is 4D.

$$f(0,2,-1) = \sqrt{0^2+2\times(2)^2+3(-1)^2}$$
$$= \sqrt{8+3} = \sqrt{11}$$

Ex 11.1
Q 2 → let $f(x,y,z) = x^2ye^{2x} + (x+y-z)^2$ find

- $f(-1,1,-1) = (-1)^2 \times 1 \times e^{2(-1)} + (-1+1-(-1))^2$
  $= e^{-2}+1$

- $\dfrac{d}{dy} f(1,y,1)$

$$f(1,y,1) = ye^2 + (1+y-1)^2$$
$$= y^2 + ye^2$$

(this is a fn of just one variable i.e y)

$$\frac{d}{dy} f(1,y,1) = \boxed{2y+e^2}$$

Q. $f(x,y) = \dfrac{xy}{x-y}$ find domain and Range.

[here we have 2 independent variables, ∴ domain is 2D] [and graph is 3D]

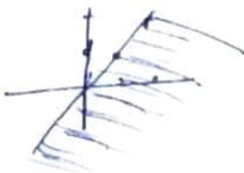
$D = \{(x,y) \; ; \; x \neq y\}$

$R = \{z \mid -\infty < z < \infty\}$. ($\because$ no restriction on values of z it can take any value) except $\pm\infty$.

11.1 Q5. $f(x,y) = \sqrt{x-y}$ find domain and Range.

2D

$D = \{(x,y) : x-y \geq 0\} = \{(x,y) : x \geq y\}$



$R = \{z : z \geq 0\}$ ∴ graph is only above xy plane. (nothing below)

11.1 Q6 $f(x,y) = \dfrac{1}{\sqrt{x-y}}$ find domain and Range.

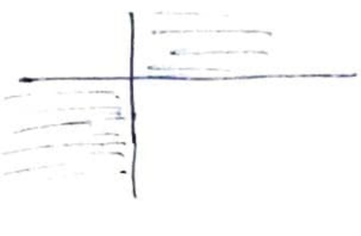
2D

$D = \{(x,y) : x > y\}$ ($\because x-y > 0$)

$R = \{z : 0 < z < \infty\}$

11.1 Q7. $f(u,v) = \sqrt{uv}$ domain is 2D graph is 3D

$D = \{(u,v) : uv \geq 0\}$



$R = \{z : 0 \leq z < \infty\}$.

1 (1) Let $f(x,y) = \sqrt{9-x^2-4y^2}$, then evaluate $f(2,1)$, $f(2t,t^2)$

(2) $f(2,1) = \sqrt{9-2^2-4(1)^2} = 1$, $f(2t,t^2) = \sqrt{9-4t^2-4(t^2)^2} = \sqrt{9-4t^2-4t^4}$

(3) domain : $\{(x,y) : x^2+4y^2 \le 9\}$ ∵ $9-x^2-4y^2 \ge 0$

(4) Range : $\{z = \sqrt{9-x^2-4y^2} : x,y \in$ domain$\}$

(5) $= \{\sqrt{9-x^2-4y^2} : 0 \le x^2+4y^2 \le 9\}$

(6) $= \{z = \sqrt{9-x^2-4y^2} : 0 \ge -x^2-4y^2 \ge -9$

(7) $= \{z = \sqrt{9-x^2-4y^2}$ i.e. $9 \ge 9-x^2-4y^2 \ge 0$

(8) = ie $3 \ge z \ge 0$

(2) Operations of fns of 2 variables

$$(f \pm g)(x,y) = f(x,y) \pm g(x,y)$$

$$\left(\frac{f}{g}\right)(x,y) = \frac{f(x,y)}{g(x,y)}, \quad g(x,y) \neq 0$$

(3) $f(x,y) = 10-x^2-y^2$

$z=1$, $10-x^2-y^2 = 1$

$\Rightarrow x^2+y^2 = 9$

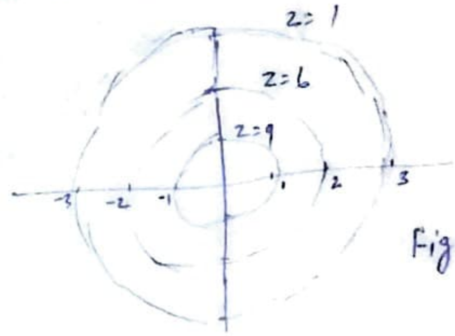


Fig A

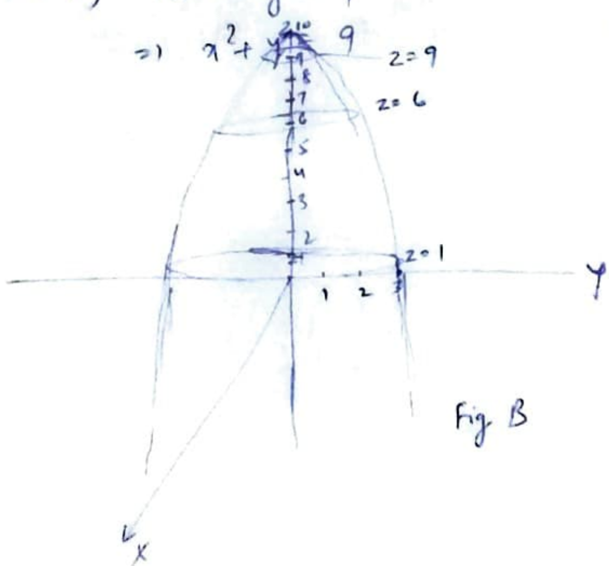


Fig B

(4) $z = x^2+y^2$

$z=C$, $x^2+y^2 = C$

$x=A$, $z = A+y^2$

$y=B$, $z = x^2+B$

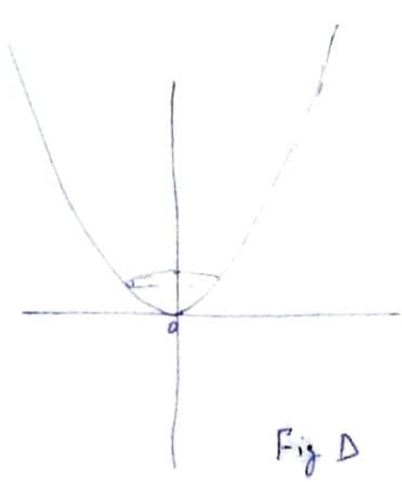


Fig D

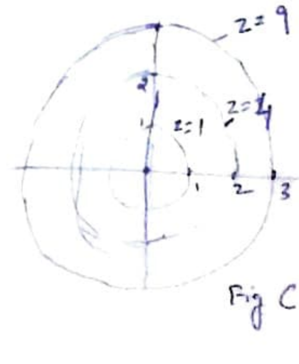


Fig C

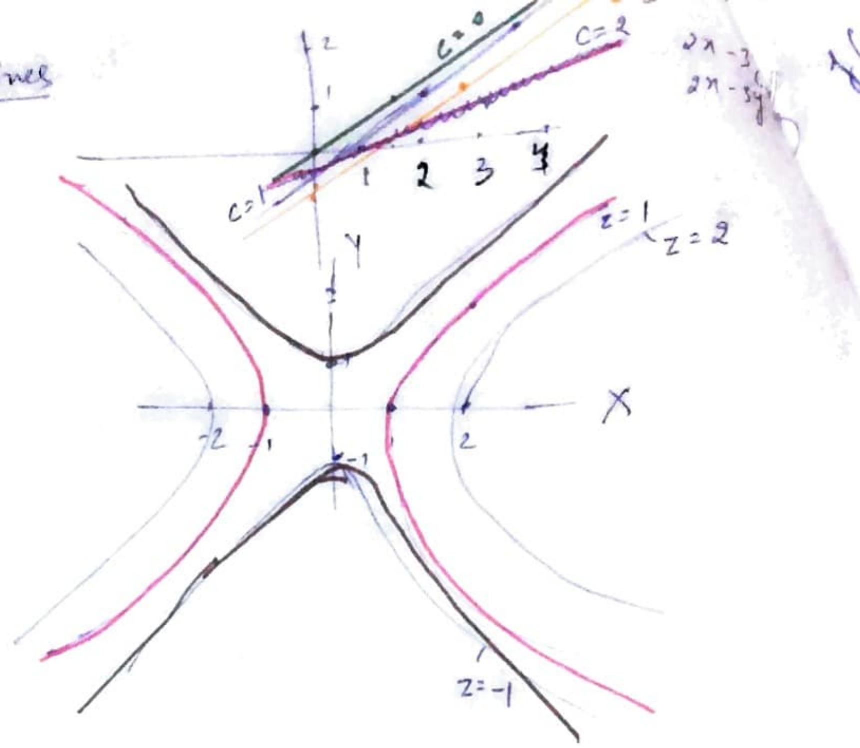
Ex

(15) $f(x,y) = 2x - 3y$

$z = C, \quad 2x - 3y = C$

$C = 0, \quad 2x - 3y = 0$

|| lines



(16) $f(x,y) = x^2 - y^2$

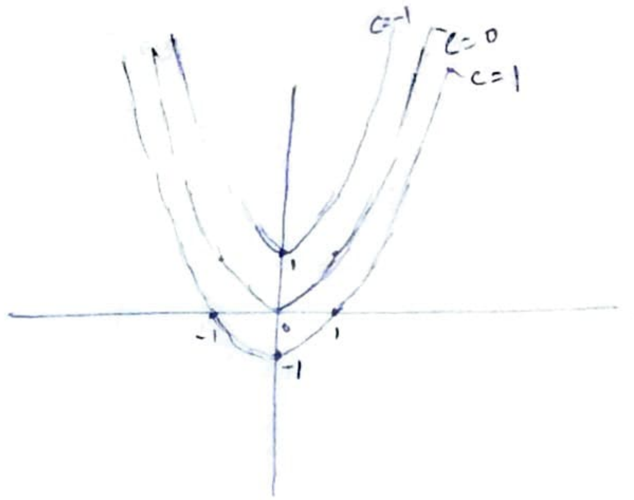
$z = C, \quad x^2 - y^2 = C$

$C = 1, \quad x^2 - y^2 = 1$

$C = 2, \quad x^2 - y^2 = 4$

$C = -1, \quad x^2 - y^2 = -1$

(18) $g(x,y) = x^2 - y$



$z = x^2 - y$

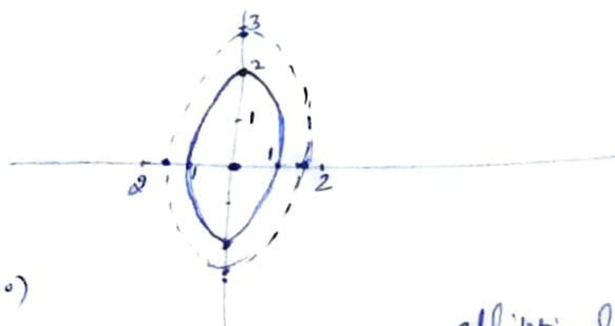
$z = C, \quad x^2 - y = C$

$C = 0, \quad x^2 = y$

$C = 1, \quad x^2 - y = 1$

$C = -1, \quad x^2 - y = -1$

(19) $f(x,y) = x^2 + \frac{y^2}{4}$



$z = x^2 + \frac{y^2}{4}$

$z = C, \quad x^2 + \frac{y^2}{4} = C$

$C = 0, \quad x^2 + \frac{y^2}{4} = 0$ single pt $(0,0)$

elliptical paraboloid.

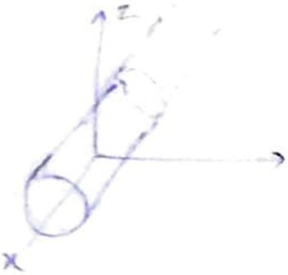
$C = 1, \quad x^2 + \frac{y^2}{4} = 1$

$C = 2 \Rightarrow x^2 + \frac{y^2}{4} = 2$

$C \geq 0$. always $\left(\because z = x^2 + \frac{y^2}{4} \geq 0\right)$

$f(x,y,z) = y^2+z^2$, $c=1$

$y^2+z^2=1$



(23) $x+y-z=1$

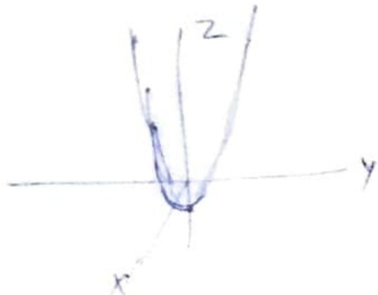
(25) $f=(x+1)^2+(y-2)^2+(z-3)^2$, $C=4$

(26) $f(x,y,z)=2x^2+2y^2-z$, $c=1$

$2x^2+2y^2-z=1$

$z=C$, $2x^2+2y^2=1+C$, $C\geq -1$

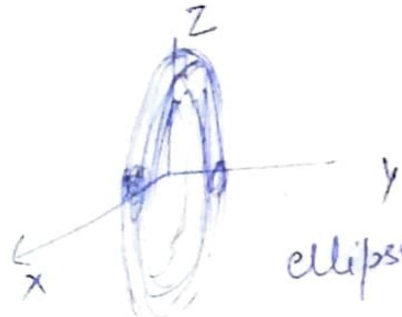
$x=A$, $2y^2-z=1-2A^2$



(27) $9x^2+4y^2+z^2=1$

$z=C$, $9x^2+4y^2=1-C^2$ $-1\leq C\leq 1$

$x=A$,



ellipsoid

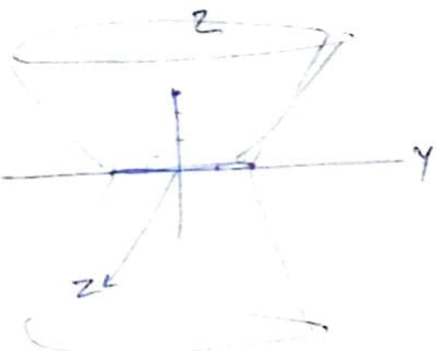
(28) $\frac{x^2}{4}+y^2+\frac{z^2}{9}=1$

(29) $\frac{x^2}{4}+\frac{y^2}{9}-z^2=1$

$z=C$, $\frac{x^2}{4}+\frac{y^2}{9}=1+c^2$ ellipse

$x=A$, $\frac{y^2}{9}-z^2=1-\frac{A^2}{4}$ hyperbola

$y=B$, $\frac{x^2}{9}-z^2=1-\frac{B^2}{9}$ hyperbola



hyperboloid of I sheet

$\left(\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{z^2}{C^2}=1\right)$

(30) $\frac{x^2}{9}-y^2-z^2=1$