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Q. If H has index 2 in a group G, Then show that H is normal in G.

Sol^n $|G:H|$ = No. of distinct left cosets of H in G

= No. of distinct right cosets of H in G.

Given that $|G:H| = 2$.

Let $x \in G$ be an arbitrary element

TS H is normal in G i.e. $xH = Hx$ $\forall x \in G$.

Case - I: When $x \in H$.

If $x \in H$, then $xH = H$ and $Hx = H$

$\therefore xH = Hx = H$

$\Rightarrow xH = Hx$. ——— (I)

Property 2
$aH = H \Leftrightarrow a \in H$

Case - II: When $x \notin H$.

$x \notin H \Rightarrow xH \neq H$ and $Hx \neq H$.

Now $|G:H| = 2$

$\Rightarrow$ No. of distinct left cosets are 2

$\Rightarrow$ H and xH are only left cosets of H in G.

$\therefore G = H \cup xH$ (Proof of Lagrange's Theorem Chapter 7.)

$\Rightarrow xH = G \setminus H$ ——— (2)

Again $|G:H| = 2$

$\Rightarrow$ No. of distinct right cosets of H in G are 2

$\Rightarrow$ H and Hx are only right cosets of H in G.

$\therefore G = H \cup Hx$

$\Rightarrow Hx = G \setminus H$ ——— (3)

From eqn (2) & (3), we have

$xH = Hx = G \setminus H$

$\Rightarrow xH = Hx$

$\therefore xH = Hx \quad \forall x \in G$

$\therefore xH = Hx$

$\therefore H$ is Normal in G.

Chapter 9 Questions:

1, 2, 3, 6, 7, 9, 10(a), 12, 13, 14, 17, 18, 20,
36, 37, 38, 41, 43, 45, 49, 50, 51, 54(a,b), 56, 57
58, 72.

Note) Factor group G/H

$$\left| G/H \right| = \frac{|G|}{|H|}$$

G/H = Set of all left cosets of H in G
= Index.