Total differential equations (Single differential equations) :—

The equation

$$P\,dx + Q\,dy + R\,dz = 0 \quad \text{——} (I)$$

is called Total differential equation or Single differential equation.

Condition of Integrability :—

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad \text{——} (II)$$

If P, Q, & R in eq^n (I) satisfies eq^n (II), we say that eq^n (I) is integrable.

Note :- Eq^n (I) is solvable iff it satisfy condition of integrability.

Example :- Solve $(y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$ — (I)

here $P = y+z$, $Q = z+x$, $R = x+y$

$$\frac{\partial P}{\partial y} = 1,\ \frac{\partial P}{\partial z} = 1,\ \frac{\partial Q}{\partial z} = 1,\ \frac{\partial Q}{\partial x} = 1,\ \frac{\partial R}{\partial x} = 1,\ \frac{\partial R}{\partial y} = 1$$

Condition of integrability

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$$= (y+z)(1-1) + (z+x)(1-1) + (x+y)(1-1)$$

$$= 0$$

hence Condition of integrability is satisfied by eq^n (I).

Now, suppose that z is a constant, ~~with~~ then $dz = 0$, and eq^n (I) becomes

$$(y+z)\,dx + (z+x)\,dy = 0 \quad \text{—— (2)}$$

After solving eq^n (2), we get

$$\cancel{(y+z)\,d}\ (y+z)(z+x) = A \quad \text{—— (3)}$$

where A is an arbitrary constant.

Now, replace 'A' with an arbitrary function of z. [as z is constant only with respect to x & y], Equation 3 become

$$(y+z)(z+x) = \phi(z) \qquad [\text{~~Rep~~}\ A=\phi(z)],$$

or $$(y+z)(z+x) - \phi(z) = 0 \quad \text{---} (4).$$

Differentiation of eqn (4) with respect to x, y, z gives

$$(y+z)dx + (z+x)dy + (x+y)dz + 2zdz - \frac{d\phi}{dz}dz = 0 \quad \text{---} (5)$$

Comparing eqn (1) & (5) we get

$$2zdz - \frac{d\phi}{dz}dz = 0$$

or

$$2zdz - d\phi = 0$$

Integration $$z^2 - \phi(z) + c^2 = 0$$

then $$\boxed{\phi(z) = z^2 + c^2}$$

Put the value of $\phi(z)$ in eqn (4) we get

$$\boxed{(y+z)(z+x) = z^2 + c^2}$$

which is soln of eqn (I).

An alternative way ~~the~~ to solve eqn (I),

$$(y+z)dx + (z+x)dy + (x+y)dz = 0$$

rewrite this

$$(x\,dy + y\,dx) + (y\,dz + z\,dy) + (z\,dx + x\,dz) = 0$$

so

$$d(xy) + d(yz) + d(zx) = 0$$

By Integration.

$$\boxed{xy + yz + zx = c}$$

Example (2)

$$\frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{1/2}} + \frac{z\,dx - x\,dz}{x^2+z^2} + 3ax^2dx + 2by\,dy + c\,dz = 0.$$

Exercise:- (1) Solve $(y+z)\,dx + dy + dz = 0$

(2) Solve $zy\,dx = zx\,dy + y^2\,dz$

(3) $(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2z\,dz = 0$

(4) $(y^2 + yz)\,dx + (xz + z^2)\,dy + (y^2 - xy)\,dz = 0$