12.4 Triple Integration

Friday, 16 October 2020 11:00 AM

→ over a rectangular box : $a \le x \le b$, $c \le y \le d$, $r \le z \le s$

$$\int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

→ over z-simple region

A z-simple region is described by
- lower bounding surface $z = u(x,y)$
- upper bounding surface $z = v(x,y)$
- projection of region on xy-plane which will be of type I or type 2.

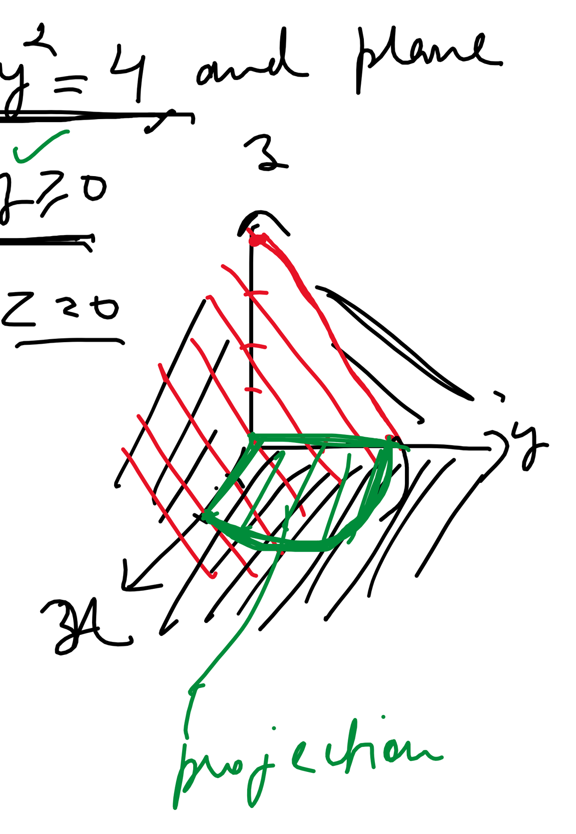
$$\iint \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\,dy\,dx \quad \text{or} \quad \iint \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\,dx\,dy$$

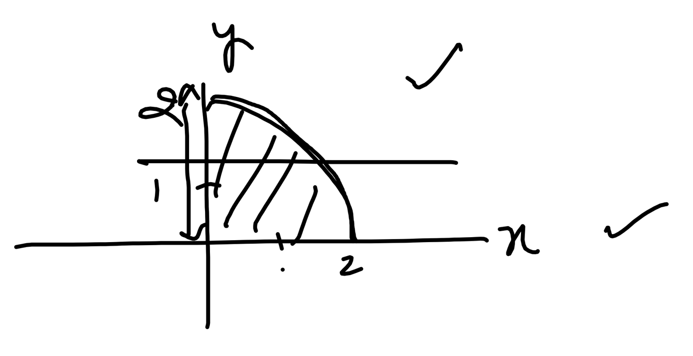
type I type 2.

Ex2 Evaluate $\iiint_D x\,dV$. D is solid in the first octant bdd by cylinder $x^2+y^2=4$ and plane $2y+z=4$, $x\ge0, y\ge0, z\ge0$

lower bdd surface xy plane ie $z=0$

upper bdd " $2y+z=4$





$x^2+y^2=4$

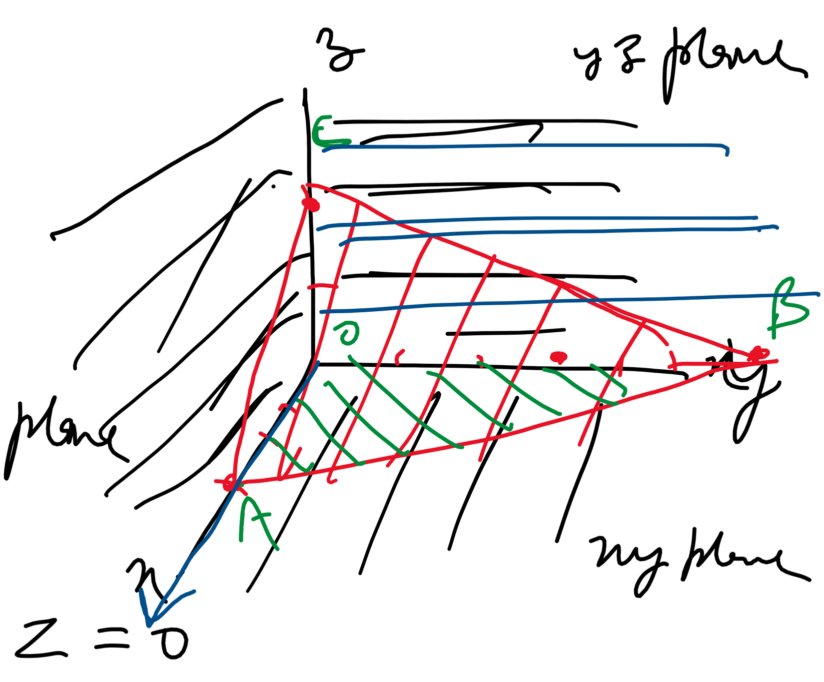
$y=\sqrt{4-x^2}$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-2y} x\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^{4-2y} x\,dz\,dx\,dy$$

$$= \frac{20}{3}$$

Ex3 Volume of Tetrahedron T bdd by plane $2x+y+3z=6$ and $x=0$, $y=0$, $z=0$

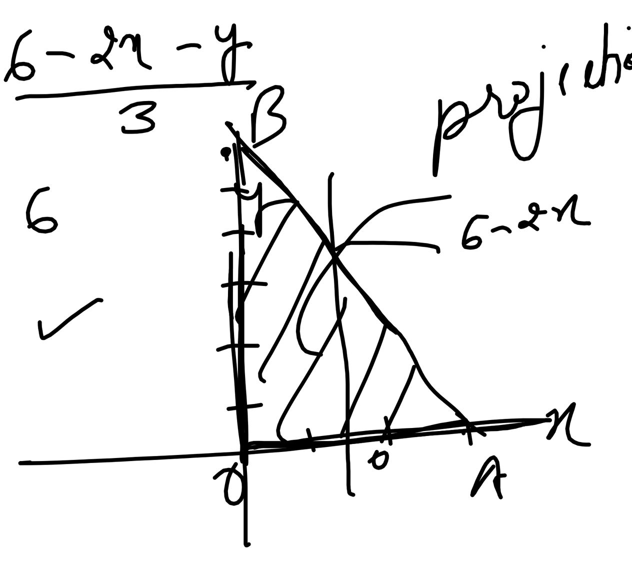


lower bdding surface $Z=0$

upper bdding surface $Z=\frac{6-2x-y}{3}$

put $z=0$ in $2x+y+3z=6$

$\Rightarrow 2x+y=6$



$$V = \int_0^3 \int_0^{6-2x} \int_0^{\frac{6-2x-y}{3}} dz\,dy\,dx$$

$$= \int_0^3 \int_0^{6-2x} \Big| z \Big|_0^{\frac{6-2x-y}{3}} dy\,dx$$

$$= \int_0^3 \int_0^{6-2x} \frac{6-2x-y}{3}\,dy\,dx$$

$$= \frac{1}{3}\int_0^3 \left(6y-2xy-\frac{y^2}{2}\right)\Big|_0^{6-2x} dx$$

$$= \frac{1}{3}\int_0^3 \left(6(6-2x)-2x(6-2x)-\frac{(6-2x)^2}{2}\right) dx$$

$$= \frac{1}{3}\int_0^3 \cancel{36}-12x-\cancel{12x}+4x^2-\frac{(\cancel{36}+4x^2-\cancel{24x})}{2}\,dx$$

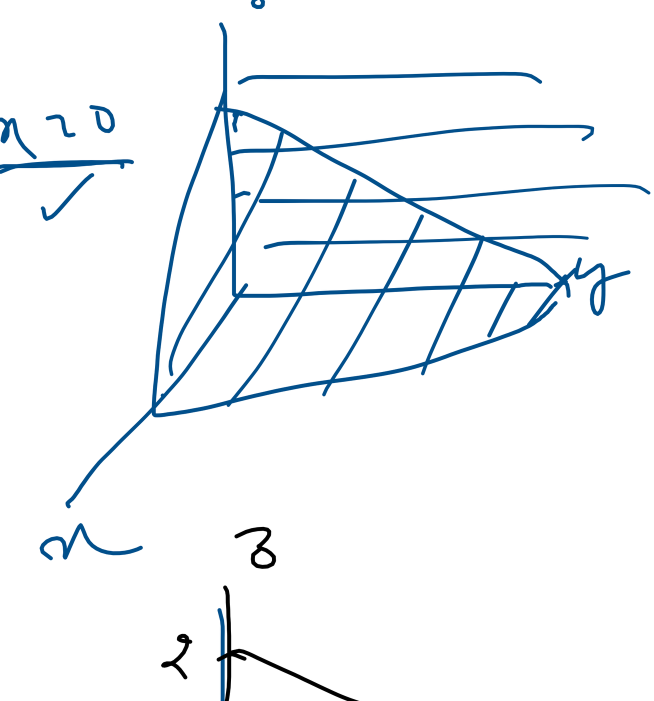
$$= \frac{1}{3}\int_0^3 (18-12x+2x^2)\,dx$$

$= 6$. cubic units.

Ex4. Find volume of T by projecting onto y-z plane (x-simple way)

$2x+y+3z=6$

x lower bdding surface : $x=0$ i.e yz plane

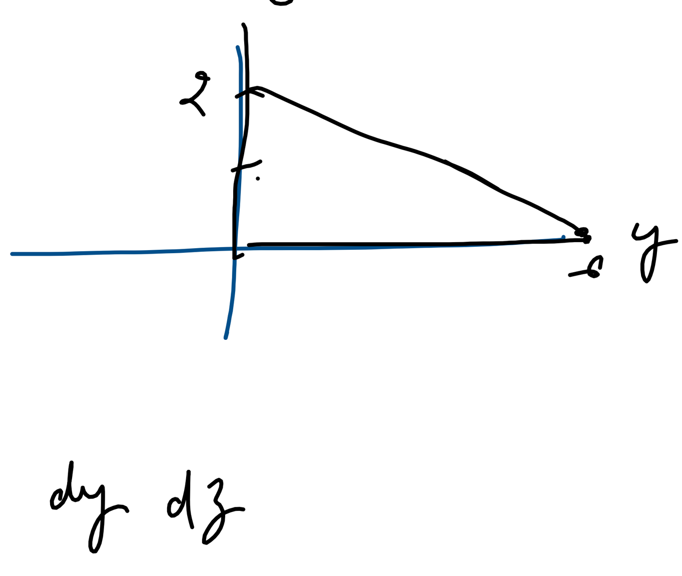


x upper bdding surface

$x=\frac{6-y-3z}{2}$

projection

$y+3z=6$, $y\ge0$, $z=0$



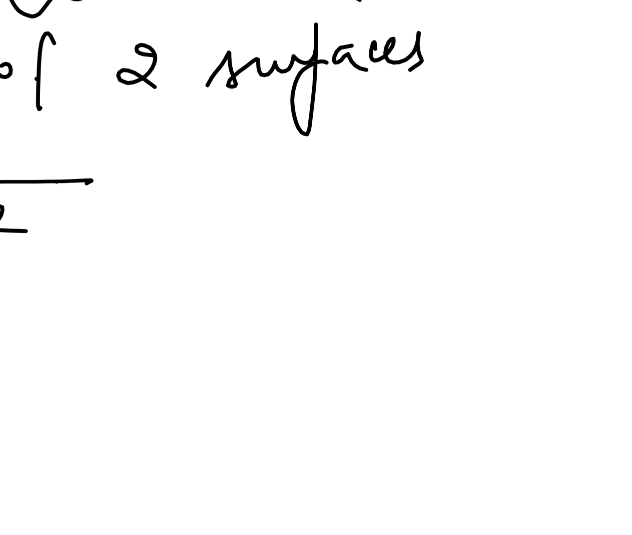
$$V = \int_0^2 \int_0^{6-3z} \int_0^{\frac{6-y-3z}{2}} dx\,dy\,dz$$

$= 6$.

Ex5 Volume of solid D bdd above by sphere $x^2+y^2+z^2=4$, below by plane $y+z=2$.

lower bding surface $z=2-y$

upper bding surface $z=\sqrt{4-x^2-y^2}$



projection : intersection of 2 surfaces

$2-y=\sqrt{4-x^2-y^2}$

$4+y^2-4y = 4-x^2-y^2$

$x^2+2y^2-4y=0$

$\Rightarrow x^2+2(y-1)^2=2$ projection

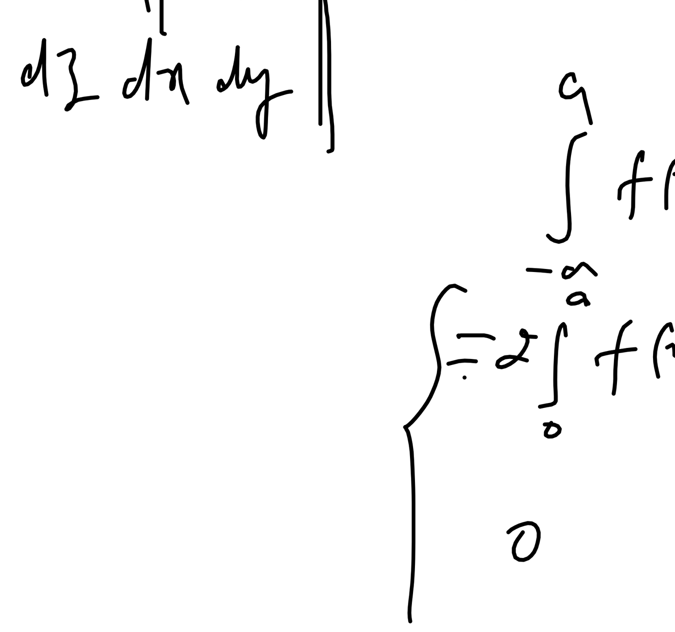
$$V = \int_0^2 \int_{-\sqrt{4y-2y^2}}^{\sqrt{4y-2y^2}} \left(\int_{2-y}^{\sqrt{4-x^2-y^2}} dz\right) dx\,dy$$

$\frac{x^2}{2}+\frac{(y-1)^2}{1}=1$

$x=0$

$y-1=\pm1$

$y=2, 0$.



$$= 2\int_0^2 \int_0^{\sqrt{4y-2y^2}} \int_{2-y}^{\sqrt{4-x^2-y^2}} dz\,dx\,dy$$

$$\int_{-a}^{a} f(x)\,dx \begin{cases} = 2\int_0^a f(x)\,dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd} \end{cases}$$