Simultaneous differential Equations:–

Before starting simultaneous differential equations, I will introduce Differential operators because this method depends on use of differential operators.

Let x be an n-times differentiable function of the independent variable t. and denote $D \equiv \frac{d}{dt}$.
then, D is a differential operator. that is

$$Dx = \frac{dx}{dt}$$

$$D^2 x = \frac{d^2 x}{dt^2}$$

$$\vdots$$

$$D^n x = \frac{d^n x}{dt^n} \quad (n = 1, 2, --)$$

further extending this operator notation, we write

$$(D+c)x = \frac{dx}{dt} + cx$$

here $D+c$ is a differential operator.

If we take $aD^m + bD^m$ as differential operator

then $(aD^n + bD^m)x = a\frac{d^n x}{dt^n} + b\frac{d^m x}{dt^n}$

If we write

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) x$$

$$= a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n x$$

Where, $a_0, a_1, \ldots, a_n$ are constants.

Then the expression

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)$$

is called a linear differential operator with constant coefficients.

Example :- Let consider the linear differential operator

$$(2D^2 + 5D - 2)$$

then

$$(2D^2 + 5D - 2) x = 2 \frac{d^2 x}{dt^2} + 5 \frac{dx}{dt} - 2x$$

if x is twice differentiable.

If we take $x = t^4$

then

$$(2D^2 + 5D - 2) t^4 = 2 \frac{d^2 t^4}{dt^2} + 5 \frac{dt^4}{dt} - 2t^4$$

$$= 24 t^2 + 20 t^3 - 2t^4.$$

If $x = \sin t$

then

$$(2D^2 + 5D - 2)\sin t = 2\frac{d^2 \sin t}{dt^2} + 5\frac{d \sin t}{dt} - 2\sin t$$

$$\boxed{(2D^2 + 5D - 2)\sin t = -2\sin t + 5\cos t - 2\sin t}$$

or If $x = e^{2t}$

then

$$(2D^2 + 5D - 2)e^{2t} = 2\frac{d^2 e^{2t}}{dt^2} + 5\frac{d e^{2t}}{dt} - 2e^{2t}$$

$$= 8e^{2t} + 10e^{2t} - 2e^{2t}$$

If $x = t^2 + 4t$

then

$$(2D^2 + 5D - 2)(t^2 + 4t) = 2\frac{d^2}{dt^2}(t^2+4t) + 5\frac{d(t^2+4t)}{dt} - 2(t^2+4t)$$

$$= 2[2+0] + 5[2t+4] - 2t^2 - 8t$$

$$= 4 + 10t + 20 - 2t^2 - 8t$$

$$= 24 + 2t - 2t^2$$

Example :- Solve

$$\left.\begin{aligned} 2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x &= t \\ 2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y &= 2 \end{aligned}\right] \quad \text{--- (I)}$$

Now, write in operator form

$$\left.\begin{aligned} (2D-3)x - 2Dy &= t \\ (2D+3)x + (2D+8)y &= 2 \end{aligned}\right] \quad \text{--- (II)}$$

Now, Apply operator $(2D+8)$ in first eq^n of (II) & operate $2D$ in second eq^n of (II).

$$(2D+8)(2D-3)x - (2D+8)(2D)y = (2D+8)t$$

$$2D(2D+3)x + (2D)(2D+8)y = (2D)(2)$$

Adding both eq^n.

$$[(2D+8)(2D-3) + 2D(2D+3)]x = (2D+8)t + (2D)2$$

$$[4D^2 + 16D - 6D - 24 + 4D^2 + 6D]x = 2 + 8t + 0$$

$$[8D^2 + 16D - 24]x = 2 + 8t$$

$$(D^2 + 2D - 3)x = t + \frac{1}{4} \quad \text{--- (3)}$$

eq^n (3) is L.D.E. with constant coefficient and sol^n of (3) is

$$\boxed{x(t) = c_1 e^{t} + c_2 e^{-3t} - \frac{t}{3} - \frac{11}{36}}$$

~~Now apply (2D+3) in first eqn of (II) &~~

Since

$$x(t) = C_1 e^{t} + C_2 e^{-3t} - \frac{t}{3} - \frac{11}{36}$$

then

$$\frac{dx}{dt} = C_1 e^{t} - 3C_2 e^{-3t} - \frac{1}{3}$$

~~Put~~ Now add both equations of (I), we get

$$4\frac{dx}{dt} + 8y = t + 2$$

or

$$8y = t + 2 - 4\frac{dx}{dt}$$

$$y = \frac{t}{8} + \frac{1}{4} - \frac{1}{2}\frac{dx}{dt}$$

Put the value of $\frac{dx}{dt}$ in this, we get

$$y(t) = \frac{t}{8} + \frac{1}{4} - \frac{1}{2}C_1 e^{t} + \frac{3}{2}C_2 e^{-3t} + \frac{1}{6}$$

$$\boxed{y(t) = -\frac{1}{2}C_1 e^{t} + \frac{3}{2}C_2 e^{-3t} + \frac{t}{8} + \frac{5}{12}}$$

Then, the general soln of (I) is

$$\boxed{\begin{aligned} x(t) &= C_1 e^{t} + C_2 e^{-3t} - \frac{t}{3} - \frac{11}{36} \\ y(t) &= -\frac{1}{2}C_1 e^{t} + \frac{3}{2}C_2 e^{-3t} + \frac{t}{8} + \frac{5}{12} \end{aligned}}$$

An other way to solve this example is, (Alternative), as we have find $x(t)$,
Now to find $y(t)$, operate $(2D+3)$ in first equation of (II) & $(2D-3)$ in second eqn of (II) we get

$$(2D+3)(2D-3)x-(2D+3)(2D)y=(2D+3)t$$

$$(2D-3)(2D+3)x+(2D-3)(2D+8)y=(2D-3)(2)$$

on substracting

$$-[(2D+3)(2D)+(2D-3)(2D+8)]y=(2D+3)t-(2D-3)(2)$$

$$-[4D^2+6D+4D^2-6D+16D-24]y=2+3t-0+6$$

$$-[8D^2+16D-24]y=3t+8$$

$$[D^2+2D-3]y=-\frac{3}{8}t-1 \quad \text{—} (4)$$

which is L.D.E with constant coefficient, we ~~have~~ can find soln

$$\boxed{y(t)=K_1e^{t}+K_2e^{-3t}+\frac{1}{8}t+\frac{5}{12}}$$

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Now we have

$$x(t) = c_1 e^{t} + c_2 e^{-3t} - \frac{t}{3} - \frac{11}{36}$$

$$y(t) = k_1 e^{t} + k_2 e^{-3t} + \frac{t}{8} + \frac{5}{12}$$

c_1 & c_2 are arbitrary Constant.

Since, order of given system of eq^n is 2.

$$\begin{vmatrix} 2D-3 & -2D \\ 2D+3 & 2D+8 \end{vmatrix} = 8D^2+16D-24$$

Therefore, the general sol^n of given system of eq^n must have only two independent Constants.

But here we have four Constants c_1, c_2, k_1 & k_2 then only two can be independent sol^n.

Now find $\frac{dx}{dt}$ & $\frac{dy}{dt}$ and put in first eq^n of (I).

$$\frac{dx}{dt} = c_1 e^{t} - 3c_2 e^{-3t} - \frac{1}{3}$$

&

$$\frac{dy}{dt} = k_1 e^{t} - 3k_2 e^{-3t} + \frac{1}{8}$$

then

$$2c_1 e^{t} - 6c_2 e^{-3t} - \frac{2}{3} - 2k_1 e^{t} + 6k_2 e^{-3t} - \frac{2}{8} - 3c_1 e^{t} \cancel{+} 3c_2 e^{-3t} + t + \frac{11}{12} = t$$

or

$$(-c_1 - 2k_1) e^{t} + (-9c_2 + 6k_2) e^{-3t} = 0$$

We get $-c_1 - 2k_1 = 0 \Rightarrow k_1 = -\frac{c_1}{2}$

$-9c_2 + 6k_2 = 0 \Rightarrow k_2 = +\frac{3}{2}c_2$

Now the general solⁿ of given system of eqⁿ is

$$x(t) = c_1 e^{t} + c_2 e^{-3t} - \frac{t}{3} - \frac{11}{36}$$

$$y(t) = -\frac{1}{2}c_1 + \frac{3}{2}c_2 e^{-3t} + \frac{t}{8} + \frac{5}{12}$$

Exercise :- Solve
$$\frac{dx}{dt} + \frac{dy}{dt} - x - 6y = e^{3t}$$

$$\frac{dx}{dt} + 2\frac{dy}{dt} - 2x - 6y = t$$

Exercises

Use the operator method described in this section to find the general solution of each of the following linear systems.

1. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x - 4y = e^t,$
   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - y = e^{4t}.$

2. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x = -2t,$
   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - y = t^2.$

3. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 3y = e^t,$
   $\dfrac{dx}{dt} + \dfrac{dy}{dt} + x = e^{3t}.$

4. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 2y = 2e^t,$
   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - 4y = e^{2t}.$

5. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = e^{-t},$
   $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = e^t.$

6. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - y = t,$
   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 4x - y = e^t.$

7. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 6y = e^{3t},$
   $\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - 2x - 6y = t.$

8. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 3y = 3t,$
   $\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - 2x - 3y = 1.$

9. $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2y = \sin t,$
   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = 0.$

10. $\dfrac{dx}{dt} - \dfrac{dy}{dt} - 2x + 4y = t,$
    $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = 1.$

11. $$2\frac{dx}{dt} + \frac{dy}{dt} + x + 5y = 4t,$$
$$\frac{dx}{dt} + \frac{dy}{dt} + 2x + 2y = 2.$$

12. $$\frac{dx}{dt} + \frac{dy}{dt} - x + 5y = t^2,$$
$$\frac{dx}{dt} + 2\frac{dy}{dt} - 2x + 4y = 2t + 1.$$

13. $$2\frac{dx}{dt} + \frac{dy}{dt} + x + y = t^2 + 4t,$$
$$\frac{dx}{dt} + \frac{dy}{dt} + 2x + 2y = 2t^2 - 2t.$$

14. $$3\frac{dx}{dt} + 2\frac{dy}{dt} - x + y = t - 1,$$
$$\frac{dx}{dt} + \frac{dy}{dt} - x = t + 2.$$

15. $$2\frac{dx}{dt} + 4\frac{dy}{dt} + x - y = 3e^t,$$
$$\frac{dx}{dt} + \frac{dy}{dt} + 2x + 2y = e^t.$$

16. $$2\frac{dx}{dt} + \frac{dy}{dt} - x - y = -2t,$$
$$\frac{dx}{dt} + \frac{dy}{dt} + x - y = t^2.$$

17. $$2\frac{dx}{dt} + \frac{dy}{dt} - x - y = 1,$$
$$\frac{dx}{dt} + \frac{dy}{dt} + 2x - y = t.$$

18. $$\frac{d^2x}{dt^2} + \frac{dy}{dt} = e^{2t},$$
$$\frac{dx}{dt} + \frac{dy}{dt} - x - y = 0.$$

19. $$\frac{d^2x}{dt^2} + \frac{dy}{dt} - x + y = 1,$$
$$\frac{d^2y}{dt^2} + \frac{dx}{dt} - x + y = 0.$$

20. $$\frac{d^2x}{dt^2} - \frac{dy}{dt} = e^t,$$
$$\frac{dx}{dt} + \frac{dy}{dt} - 4x - y = 2e^t.$$

21. $$\frac{d^2x}{dt^2} - \frac{dy}{dt} = t + 1,$$
$$\frac{dx}{dt} + \frac{dy}{dt} - 3x + y = 2t - 1.$$

22. $$\frac{d^2x}{dt^2} + 4\frac{dy}{dt} + x - 4y = 0,$$
$$\frac{dx}{dt} + \frac{dy}{dt} - x + 9y = e^{2t}.$$

In each of Exercises 23–26, transform the single linear differential equation of the form (7.6) into a system of first-order differential equations of the form (7.9).

23. $$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = t^2.$$

24. $$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = e^{3t}.$$

25. $$\frac{d^3x}{dt^3} + t\frac{d^2x}{dt^2} + 2t^3\frac{dx}{dt} - 5t^4 = 0.$$

26. $$\frac{d^4x}{dt^4} - t^2\frac{d^2x}{dt^2} + 2tx = \cos t.$$

Ex. 2. Solve $\left.\begin{aligned} \frac{dx}{dt} - 7x + y &= 0 \\ \frac{dy}{dt} - 2x - 5y &= 0 \end{aligned}\right\}$

Ex. 3. Solve $\left.\begin{aligned} \frac{dx}{dt} + 2x - 3y &= t \\ \frac{dy}{dt} - 3x + 2y &= e^{2t} \end{aligned}\right\}$

Ex. 4. Solve $\left.\begin{aligned} 4\frac{dx}{dt} + 9\frac{dy}{dt} + 44x + 49y &= t \\ 3\frac{dx}{dt} + 7\frac{dy}{dt} + 34x + 38y &= e^{t} \end{aligned}\right\}$

Ex. 5. Solve $\left.\begin{aligned} \frac{d^2x}{dt^2} - 3x - 4y &= 0 \\ \frac{d^2y}{dt^2} + x + y &= 0 \end{aligned}\right\}$