23 April 2021
11:12

$$p_{ij}^{n} = P\{X_n = j \mid X_0 = i\}$$

state i is recurrent if $\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$

and state i is transient if $\sum_{n=1}^{\infty} P_{ii}^{n} < \infty$

All states of the same class are of same type.

All states of a finite irreducible Markov chain are recurrent.

Random Walk:

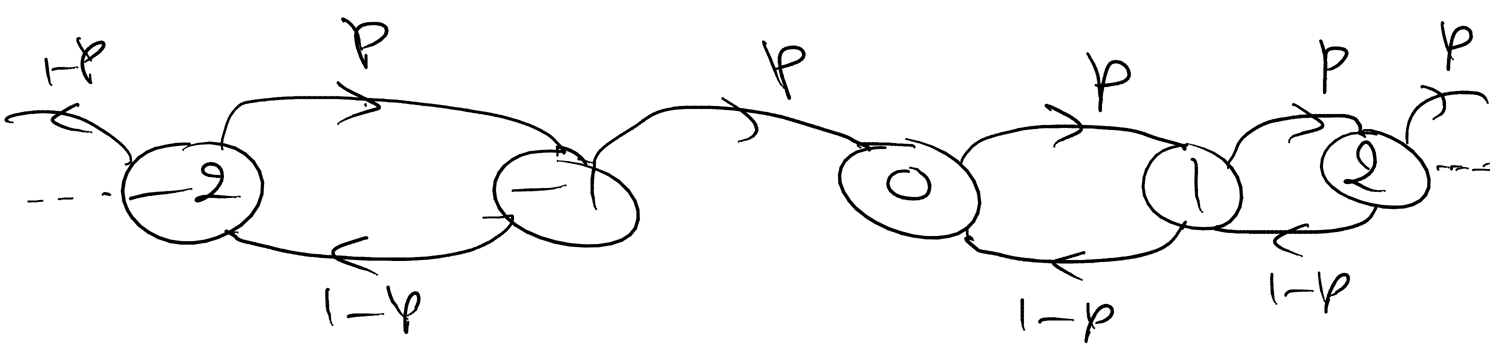
Consider a Markov chain whose state space consists of the integers $i = 0, \pm 1, \pm 2, \ldots$ and have transition probabilities given by

$$P_{i,i+1} = p$$

$$P_{i,i-1} = 1-p \quad , \; 0 < p < 1$$

$$P = \begin{array}{c|c} & \cdots \; -3 \;\; -2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \; \cdots \\ \begin{matrix} \vdots \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{bmatrix} & & & & & & & \\ \cdots & 1-p & 0 & p & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1-p & 0 & p & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1-p & 0 & p & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1-p & 0 & p & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1-p & 0 & p & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \end{array}$$

Transition Diagram



all states are clearly communicate

$\Rightarrow$ all states are in same class

or this markov chain is irreducible.

$\Rightarrow$ All states are either transient

or recurrent.

Let us consider state 0.

$$\sum_{n=1}^{\infty} P_{00}^{n} = ?$$

Now, we can easily see that it is impossible to be even after an odd number of plays. i.e. it is impossible to return on 0 in odd steps

$$, \; P^{2n-1} = 0 \qquad n = 1, 2 \quad \text{———} \; (1)$$

$$\therefore \; P_{00} = 0, \quad n = 1, 2, \ldots \text{ --- } (1)$$

Now, after $2n$ steps if we move right by n steps and move left by n steps with probabilities p and $1-p$ respectively

$$\therefore \; P_{00}^{2n} = \binom{2n}{n} p^n (1-p)^n$$

$$\Rightarrow \; P_{00}^{2n} = \frac{(2n)!}{n!\, n!} \big(p(1-p)\big)^n, \quad n = 1, 2, 3 \ldots \text{ --- } (2)$$

$$\therefore \sum_{n=1}^{\infty} P_{00}^{n} = \sum_{n=1}^{\infty} P_{00}^{2n} \qquad (\text{from eq}^n (1) \,\&\, (2))$$

$$\Rightarrow \sum_{n=1}^{\infty} P_{00}^{n} = \sum_{n=1}^{\infty} \frac{(2n)!}{n!\, n!} \big(p(1-p)\big)^n \text{ --- } (3)$$

Now, Stirling's approximation gives

$$n! \simeq n^{n+1/2} e^{-n} \sqrt{2\pi}$$

$$\therefore \binom{2n}{n} = \frac{2n!}{n!\, n!}$$

$$= \frac{(2n)^{2n+1/2} e^{-2n} \cdot \sqrt{2\pi}}{\left\{ n^{n+1/2} e^{-n} \sqrt{2\pi} \right\}^2}$$

$$= \frac{(2n)^{2n+1/2} e^{-2n} \sqrt{2\pi}}{n^{2n+1} e^{-2n} (2\pi)}$$

$$= \frac{(2)^{2n+1/2} \cdot n^{2n+1/2}}{n^{2n+1} \sqrt{2\pi}}$$

$$= \frac{(4^n) \sqrt{2}}{\quad} = \frac{4^n}{\quad}$$