Solving a differential equation by reduction of order

Theorem:- Let f be a non-trivial solution of the nth order homogeneous linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad (1)$$

Then, the transformation $y = v f(x)$ reduces equation (1) to an $(n-1)$st order homogeneous equation in the dependent variable $w = \frac{dv}{dx}$.

We will verify the theorem for second order linear differential equation.

Suppose $f(x)$ is a known non trivial solution of the second-order homogeneous linear equation

$$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad (2)$$

$[a_0(x) \neq 0]$

Let take the transformation $y = v f(x)$ — (3)

where f is the known solution of (2) and v is a function of x that will be determined.

Then, differentiating (3) with respect to 'x', we obtain

$$\frac{dy}{dx} = v f'(x) + f(x) \frac{dv}{dx} \quad \text{--- (4)}$$

&

$$\frac{d^2y}{dx^2} = f(x) \frac{d^2v}{dx^2} + 2 f'(x) \frac{dv}{dx} + f''(x) v \quad \text{--- (5)}$$

Substituting (3), (4) & (5) into equation (2), we get

$$a_0(x)\left[f(x)\frac{d^2v}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v\right] + a_1(x)\left[f(x)\frac{dv}{dx} + v f'(x)\right] + a_2(x) f(x) v = 0$$

or

$$a_0(x) f(x) \frac{d^2v}{dx^2} + [2a_0 f'(x) + a_1(x) f(x)]\frac{dv}{dx} + [a_0(x) f''(x) + a_1(x) f'(x) + a_2 f(x)]v = 0. \quad \text{--- (6)}$$

Since f is a solution of (2) the coefficient of v

$a_0(x) f''(x) + a_1(x) f'(x) + a_2 f(x) = 0$.

Therefore eqn (6) become

$$a_0(x) f(x) \frac{d^2v}{dx^2} + [2a_0 f'(x) + a_1(x) f(x)]\frac{dv}{dx} = 0 \quad \text{--- (7)}$$

Let $w = \frac{du}{dx}$, $\Rightarrow \frac{dw}{dx} = \frac{d^2u}{dx^2}$,

then eqn (7) reduces to

$$a_0(x) f(x) \frac{dw}{dx} + [2a_0 f'(x) + a_1(x) f(x)] w = 0 \quad \text{--- (8)}$$

Equation (8) is first order homogeneous linear differential equation in the dependent variable w. Equation (8) can be written as

$$\frac{dw}{w} = -\left[2\frac{f'(x)}{f(x)} + \frac{a_1(x)}{a_0(x)}\right] dx \quad \text{--- (9)}$$

{as $f(x) \neq 0$ and $a_0(x) \neq 0$}.

Integrating eqn (9), we get

$$\log|w| = -2\log|f(x)| - \int \frac{a_1(x)}{a_0(x)} dx + \log|C_1|$$

or $$\log|w| = -\log[f(x)]^2 - \int \frac{a_1(x)}{a_0(x)} dx + \log|C_1|$$

or $$\boxed{w = \frac{C_1 e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2}} \quad \text{--- (10)}$$

This is general solution of equation (8).

Now, using the transformation

$$\frac{du}{dx} = w \text{ , we get}$$

$$\frac{du}{dx} = \frac{c_1 e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2}$$

By Integration, we obtain

$$u = \int \frac{c_1 e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2} dx + c_2 \quad \text{—(11)}$$

finally from eqn (3), we get

$$y(x) = f(x) \left[\int \frac{c_1 e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2} dx + c_2 \right]$$

or

$$y(x) = c_1 f(x) \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2} dx + c_2 f(x)$$

or

$y(x) = c_1 g(x) + c_2 f(x)$ is the general solution of eqn (2). ~~and~~ where

$$g(x) = f(x) \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2} dx \text{ is}$$

~~an~~ other Linearly independent solⁿ of (2),

f & g are Linearly independent.

Example:- Given that

$y = x$ is a solution of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \qquad \text{--- (1)}$$

find a linearly independent solution of (1) by reducing the order.

Solⁿ:- Let $y = xu \qquad \text{--- (2)}$

then $\frac{dy}{dx} = x\frac{du}{dx} + u$ and $\frac{d^2y}{dx^2} = x\frac{d^2u}{dx^2} + 2\frac{du}{dx}$

Substituting the values of y, $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in eqⁿ (1) we get

$$(x^2+1)\left[x\frac{d^2u}{dx^2} + 2\frac{du}{dx}\right] - 2x\left[x\frac{du}{dx} + u\right] + 2xu = 0$$

or

$$x(x^2+1)\frac{d^2u}{dx^2} + 2(x^2+1)\frac{du}{dx} - 2x^2\frac{du}{dx} \cancel{-2xu} + \cancel{2xu} = 0$$

or

$$x(x^2+1)\frac{d^2u}{dx^2} + 2\frac{du}{dx} = 0 \qquad \text{--- (3)}$$

Let $w = \frac{du}{dx}$ or $\frac{dw}{dx} = \frac{d^2u}{dx^2}$

eqⁿ (3), becomes

$$x(x^2+1)\frac{dw}{dx} + 2w = 0 \qquad \text{--- (4)}$$

which is first order L.D.E. (Homogeneous).

$$\frac{dw}{w} = \frac{-2}{x(x^2+1)}\,dx$$

or

$$\frac{dw}{w} = \left(-\frac{2}{x} + \frac{2x}{x^2+1}\right) dx$$

Integrating, we get

$$\log w = -2\log x + \log(x^2+1) + \log c_1$$

or

$$\boxed{w = \frac{c_1(x^2+1)}{x^2}} \quad \text{--- (5)}$$

which is general soln of eqn (4),

we have $$\frac{dv}{dx} = w$$

then

$$\frac{dv}{dx} = c_1\frac{(x^2+1)}{x^2}$$

or

$$dv = c_1\left(1+\frac{1}{x^2}\right) dx$$

By Integrating $$\boxed{v = c_1\left(x - \frac{1}{x}\right) + c_2}$$

then eqn (2) gives $$y = x\left[c_1\left(x-\frac{1}{x}\right)+c_2\right]$$

$$\boxed{y = c_1(x^2-1) + c_2 x}$$

which is general soln of (1) and (x^2-1) is other L. linear Independent soln of (1).

Exercises

1. Given that $y = x$ is a solution of

$$x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 4y = 0,$$

find a linearly independent solution by reducing the order. Write the general solution.

2. Given that $y = x + 1$ is a solution of

$$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1) \frac{dy}{dx} + 3y = 0,$$

find a linearly independent solution by reducing the order. Write the general solution.

3. Given that $y = x$ is a solution of

$$(x^2 - 1) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0,$$

find a linearly independent solution by reducing the order. Write the general solution.

4. Given that $y = x$ is a solution of

$$(x^2 - x + 1) \frac{d^2y}{dx^2} - (x^2 + x) \frac{dy}{dx} + (x+1)y = 0,$$

find a linearly independent solution by reducing the order. Write the general solution.

5. Given that $y = e^{2x}$ is a solution of

$$(2x+1) \frac{d^2y}{dx^2} - 4(x+1) \frac{dy}{dx} + 4y = 0,$$

find a linearly independent solution by reducing the order. Write the general solution.

6. Given that $y = x^2$ is a solution of

$$(x^3 - x^2) \frac{d^2y}{dx^2} - (x^3 + 2x^2 - 2x) \frac{dy}{dx} + (2x^2 + 2x - 2)y = 0,$$

find a linearly independent solution by reducing the order. Write the general solution.