Group Action :-

A gp action of a gp G on a set A is a map from $G \times A$ to A (written as $g \cdot a$) satisfying the following properties:

(i) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a \quad \forall g_1, g_2 \in G$ & $a \in A$

(ii) $1 \cdot a = a \quad \forall a \in A$.

Ex:- (i) $G \times A \to A$
$g \cdot a = a$ trivial action.

(ii) $F \times V \to V$.

$\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$\alpha (r_1, r_2, -, r_n) = (\alpha r_1, \alpha r_2, \dots, \alpha r_n)$

$\alpha \in \mathbb{R}, \quad (r_1, r_2, -, r_n) \in \mathbb{R}^n$.

Let G acts on a set A. for each fixed $g \in G$ we get a mapping.

$\sigma_g : G \to A$

as $\sigma_g(a) = g \cdot a$

(i) for each fixed $g \in G$, σ_g is a permutation of A.
(ii) the map from G to S_A defined by $g \mapsto \sigma_g$ is a homomorphism.

$S_A \to$ set of all permutation of A.

Proof:- (i) σ_g is permutation

$g \in G$, $\exists\, g^{-1} \in G$ define $\sigma_{g^{-1}} : A \to A$ as
$\sigma_{g^{-1}}(a) = g^{-1}.a$

then

$$(\sigma_g \circ \sigma_{g^{-1}})(a) = \sigma_g(\sigma_{g^{-1}}(a))$$
$$= \sigma_g(g^{-1}.a) = g(g^{-1}.a)$$
$$= (g.g^{-1}).a = a$$

& $$(\sigma_{g^{-1}} \circ \sigma_g)(a) = \sigma_{g^{-1}}(\sigma_g(a))$$
$$= \sigma_{g^{-1}}(g.a) = g^{-1}(g.a)$$
$$= (g^{-1}g).a = a.$$

$\Rightarrow$ $\sigma_{g^{-1}}$ is inverse of σ_g $\Rightarrow$ σ_g is permutation.

(ii) $\phi : G \to S_A$ as
$\phi(g) = \sigma_g$

Claim :- ϕ is homomorphism.

i.e. $\phi(g_1 g_2) = \phi(g_1) \circ \phi(g_2)$

Let $\phi(g_1 g_2)(a) = (g_1 g_2) \cdot a$

$= g_1 \cdot (g_2 \cdot a)$

$= g_1 \cdot (\sigma_{g_2})$

$= (\sigma_{g_1} \circ \sigma_{g_2})(a)$

$= (\phi(g_1) \circ \phi(g_2))(a)$

a is arbitrary.

$\Rightarrow \phi(g_1 g_2) = \phi(g_1) \circ \phi(g_2)$

$\Rightarrow$ ϕ is a homomorphism.

Here ϕ is called the permutation representation associated to the given action

Examples:-

(i) G be gp and $A \to$ non-empty set.

$g \cdot a = a \quad \forall g \in G, \ a \in A$

then distinct elements of g induce the same permutation on A (i.e. identity permutation)

and the associated permutation representation $G \to S_A$ is trivial homomorphism.