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1. LEARNING OUTCOMES

This chapter will introduce the reader to the concept of metrics (a class of functions which is regarded as generalization of the notion of distance) and metric spaces. A lot emphasis has been given to motivate the ideas under discussion to help the reader develop skill in using his imagination to visualize the abstract nature of the subject. Variety of examples along with real life applications have been provided to understand and appreciate the beauty of metric spaces. Moreover the concepts of metric subspace, metric superspace, isometry (i.e., distance preserving functions between metric spaces) and norms on linear spaces are also discussed in detail.

2. PREREQUISITES

It is assumed that the reader has done a course which includes introductory real analysis, that is, the reader has familiarity with concepts like convergence of sequence of real numbers, continuity of real valued functions etc. But it is nowhere assumed that the reader has mastered these topics and hence all the concepts are well explained. Next we list few inequalities that are required in the chapter.

Inequalities

1. **Cauchy-Schwarz Inequality**

Let $x_i, y_i \in \mathbb{R}$ for $i = 1,2,\ldots,n$, then following inequality holds:

$$\sum_{i=1}^{n}|x_i y_i| \leq \sqrt{\sum_{i=1}^{n}|x_i|^2}\sqrt{\sum_{i=1}^{n}|y_i|^2}\ .$$

2. **Minkowski's Inequality**

Let $x_i, y_i \in \mathbb{R}$ for $i = 1,2,\ldots,n$ and $p \geq 1$ be any real number. Then

$$\left(\sum_{i=1}^{n}|x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}\ .$$

3. **Minkowski's Inequality for Infinite Sums**

Let $p \geq 1$ be any real number and $\{x_n\}_{n\geq1}$, $\{y_n\}_{n\geq1}$ be real sequences such that

$$\sum_{n=1}^{\infty}|x_n|^p < \infty \ \ and \sum_{n=1}^{\infty}|y_n|^p < \infty\ .$$

Then $\sum_{n=1}^{\infty}|x_n + y_n|^p$ is convergent. Moreover,

$$\left(\sum_{n=1}^{\infty}|x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}|y_n|^p\right)^{\frac{1}{p}}\ .$$

Theorem A. For any $w, x, y, z \in \mathbb{R}$,

$$\left[\sqrt{(w^2 + y^2)} + \sqrt{(x^2 + z^2)}\right]^2 \geq [w + x]^2 + [y + z]^2\ .$$

Proof: Consider

$$(wz - xy)^2 \geq 0$$

$$\Rightarrow \quad w^2z^2 + x^2y^2 - 2wzxy \geq 0$$

$$\Rightarrow \quad w^2z^2 + x^2y^2 \geq 2wzxy$$

$$\Rightarrow \quad w^2x^2 + y^2z^2 + w^2z^2 + x^2y^2 \geq w^2x^2 + y^2z^2 + 2wzxy \qquad [adding\ w^2x^2 + y^2z^2\ both\ sides\ ]$$

$$\Rightarrow \quad w^2(x^2 + z^2) + y^2(x^2 + z^2) \geq (wx + yz)^2$$

$$\Rightarrow \quad (w^2 + y^2)(x^2 + z^2) \geq (wx + yz)^2$$

$$\Rightarrow \quad \sqrt{(w^2 + y^2)(x^2 + z^2)} \geq (wx + yz) \qquad [\textbf{\textit{This is Cauchy Schawrz Inequality}}]$$

$$\Rightarrow \quad 2\sqrt{(w^2 + y^2)(x^2 + z^2)} \geq 2(wx + yz)$$

$$\Rightarrow \quad (w^2 + y^2) + (x^2 + z^2) + 2\sqrt{(w^2 + y^2)(x^2 + z^2)} \geq (w^2 + y^2) + (x^2 + z^2) + 2(wx + yz)$$

$$\Rightarrow \quad \left[\sqrt{(w^2 + y^2)} + \sqrt{(x^2 + z^2)}\right]^2 \geq [w + x]^2 + [y + z]^2$$

Hence the Inequality. ■

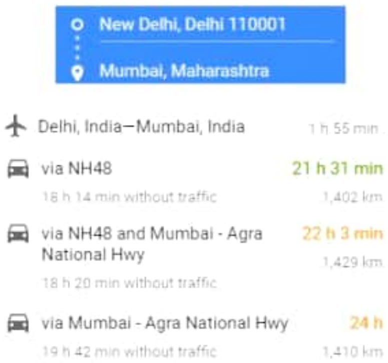
3. INTRODUCTION

A metric space is a non-empty set equipped with structure determined by a well-defined notion of distance. The term 'metric' is derived from the word metor (measure). Natural and immediate questions that comes to mind are what do we mean by measure, what can be measured and how it can be measured? In the search of answers to these questions, let us consider the following example:

Suppose a person wants to go from New Delhi to Mumbai. The adjoining figure gives possible routes from New Delhi to Mumbai. Depending on the situation, he may travel by taking any of the given possible option. We note that there are two different ways to interpret his journey.

(i) Navigational distance (in km) from New Delhi to Mumbai.

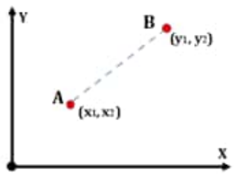
(ii) Navigational time (in hrs) to reach from New Delhi to Mumbai



Suppose he travels via NH48, then the distance travelled is 1402 km and time taken is 21h 31min. So in this example, Time and Distance represent two different modes of measurement.

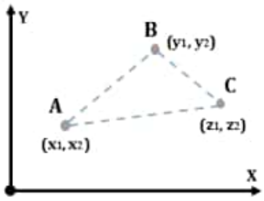
Here, we shall discuss and learn about a very special class of functions that 'measure difference' which mathematicians were able to identify in the beginning of the 20^{th} century. In the mathematical literature, this special class is represented as "distance". In the plane, distance between two points is measured along the straight line joining them. Our objective in this chapter is to illustrate through examples the different ways of measuring difference (distance) between objects besides straight line measurements, so that students can grasp the abstract nature of the subject.

To begin with, let us observe the fundamental properties of straight line distance measured between two points in $\mathbb{R}^2$. From high school geometry, we know that straight line distance between points A and B is $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.



Properties of straight line distance

1. Measurement between distinct points is a positive real number.
2. Two points in a space are identical if and only if measurement between them is zero.
3. Measurement is symmetric in nature i.e., distance measured along A to B is same as measured along B to A.
4. Measurement between two points is less than or equal to the total distance taken when we travel via some other point.



$AC \leq AB + BC$

From the first two properties, we observe that straight line distance is non-negative real number.

How to generalize all these ideas under one notion so that the properties remain intact? The solution is provided by real valued functions which measures difference. Such functions are known as **metric** in the mathematical literature. Further since the prototype for such functions is straight line distance, these functions are often regarded as **distance functions**.

These functions were first considered in 1905, by the French mathematician Maurice Frechet who thought of generalizing the notion of distances and extending them to arbitrary sets. In his doctoral dissertation "*Less Espaces Abstrait*", he introduced the concept of a metric on a set.

Metric Space

Let X be any set and let $d: X \times X \to \mathbb{R}$ be a real valued function satisfying the following properties:

P1. $d(x, y) \geq 0$ for all $x, y \in X$;

P2. $d(x, y) = 0 \iff x = y$

P3. $d(x, y) = d(y, x)$ for all $x, y \in X$

P4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

The function d is called a **metric** on X (sometimes the **distance function** on X). The ordered pair (X, d) is called a **metric space**. Thus a metric space consists of a non-empty set equipped with a concept of distance (metric). If there is no ambiguity on the metric considered, then we simply denote the metric space (X, d) by X. We refer the elements in X as points and $d(x, y)$ as the distance between the points x and y.

Trivially, an empty function is the only metric on the empty set. Also, owing to condition second, the only metric on a singleton set is the zero function.

4. EXAMPLES OF METRIC SPACES

Example 4.1 The Real Line $\mathbb{R}$

Let $\mathbb{R}$ be the set of all real numbers and $u: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined as

$$u(x,y) = |x-y| \quad \forall\ x,y \in \mathbb{R}.$$

Then we shall prove that u is a **metric** on $\mathbb{R}$

First observe that by definition, $(x,y) \geq 0\ \forall\, x,y \in \mathbb{R}$. Therefore **P1** holds.

For any x,y in $\mathbb{R}$,

$$u(x,y) = 0 \Leftrightarrow |x-y| = 0 \Leftrightarrow x = y.$$

Therefore **P2** holds.

Again, for any x,y in $\mathbb{R}$,

$$u(x,y) = |x-y| = |y-x| = u(y,x).$$

Therefore **P3** holds.

To see the triangle inequality (**P4**), suppose $x,y,z \in \mathbb{R}$ be any three points.

Consider

$$\begin{aligned} u(x,y) &= |x-y| \\ &= |(x-z)+(z-y)| \\ &\leq |x-z| + |z-y| \\ &= u(x,z) + u(z,y). \end{aligned}$$

It follows that

$$u(x,y) \leq\ u(x,z) + u(z,y) \quad \forall\ \ x,y,z \in \mathbb{R}.$$

Thus all the four axioms are satisfied. Hence u is a metric on $\mathbb{R}$ and the ordered pair $(\mathbb{R}, u)$ is a metric space. The metric u is called the **usual or standard metric or Euclidean metric** on $\mathbb{R}$. ■

Example 4.2 The Euclidean Metric on $\mathbb{C}$ (Extension of Euclidean metric on $\mathbb{R}$)

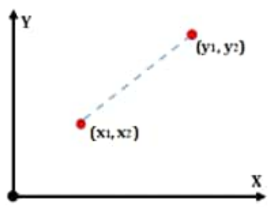
Let $\mathbb{C}$ be the set of all complex number and $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be a function defined as

$$d(z,z') = |z - z'| \quad \forall\ z,z' \in \mathbb{C}\ .$$

Then d is a metric on $\mathbb{C}$, called the **usual metric or Euclidean Metric on** $\mathbb{C}$. Of course, d is an extension to $\mathbb{C} \times \mathbb{C}$ of the Euclidean metric u on $\mathbb{R}$ i.e.,

$$u = d|_{\mathbb{R}}\ . \qquad ■$$

Example 4.3 The Euclidean Plane $\mathbb{R}^2$



Let $X = \mathbb{R}^2$ be the set of all ordered pairs of real numbers and $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined as

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \quad \forall\, x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2.$$

We shall show that d is a metric on $\mathbb{R}^2$. By definition,

$$d(x,y) \geq\ 0 \ \ \forall\ \ x,y \in \mathbb{R}^2.$$

For any $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$,

$$\begin{aligned} d(x,y) = 0 \quad &\Leftrightarrow \quad \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0 \\ &\Leftrightarrow \quad x_1 - y_1 = 0 \quad and \quad x_2 - y_2 = 0 \\ &\Leftrightarrow \quad x_1 = y_1 \; and \;\; x_2 = y_2 \\ &\Leftrightarrow \quad x = y. \end{aligned}$$

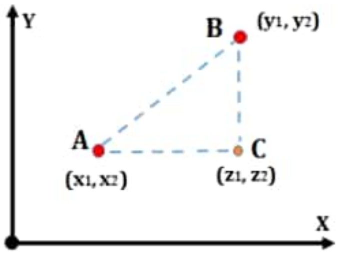
For all $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$,

$$\begin{aligned} d(x,y) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \\ &= d(y,x). \end{aligned}$$

Suppose $x = (x_1, x_2),\ y = (y_1, y_2),\ z = (z_1, z_2) \in \mathbb{R}^2$ be any three points.

Consider,

$$\begin{aligned} d(x,y) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= \sqrt{[(x_1 - z_1) + (z_1 - y_1)]^2 + [(x_2 - z_2) + (z_2 - y_2)]^2} \\ &= \sqrt{[a + b]^2 + [c + d]^2} \end{aligned}$$



where $= x_1 - z_1,\ \ b = z_1 - y_1,\ \ c = x_2 - z_2 \ \ and \ \ d = z_2 - y_2$.

Applying **Theorem A** , we get

$$\begin{aligned} d(x,y) &\le \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2} \\ &\le \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2} + \sqrt{(z_1 - y_1)^2 + (z_2 - y_2)^2} \\ &\le d(x,z) + d(z,y). \end{aligned}$$

Thus all the four axioms are satisfied. It follows that d is a metric on $\mathbb{R}^2$ and the ordered pair $(\mathbb{R}^2, d)$ is a metric space. The metric d is called the **Euclidean metric** on $\mathbb{R}^2$, and the metric space $(\mathbb{R}^2, d)$ is called the **2-dimensional Euclidean Space** $\mathbb{R}^2$. ■

Example 4.4 Taxi Cab Metric on $\mathbb{R}^2$

Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers and $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined as

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \qquad \forall\ x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

We shall show that d is a metric on $\mathbb{R}^2$.

By definition, d is a non-negative function and hence **P1** holds.

For **P2**, consider any $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$, then

$$\begin{aligned} d(x,y) = 0 &\Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0 \\ &\Leftrightarrow |x_1 - y_1| = 0 \ \ and \ \ |x_2 - y_2| = 0 \\ &\Leftrightarrow x_1 = y_1 \ \ and \ \ x_2 = y_2 \\ &\Leftrightarrow (x_1, x_2) = (y_1, y_2) \ \ i.e., \qquad x = y\ . \end{aligned}$$

Now again for **P3,** consider any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$,

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(y,x)\,.$$

Thus **P3** is satisfied.

To see triangle inequality **(P4),** let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in \mathbb{R}^2$ be any points in $\mathbb{R}^3$. Then

$$\begin{aligned} d(x, y) &= |x_1 - y_1| + |x_2 - y_2| \\ &= |(x_1 - z_1) + (z_1 - y_1)| + |(x_2 - z_2) + (z_2 - y_2)| \\ &\le |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\ &= |x_1 - z_1| + |x_2 - z_2| + |z_1 - y_1| + |z_2 - y_2| \\ &= d(x, z) + d(z, y) \ . \end{aligned}$$

Hence all the four axioms of a metric is satisfied by d, therefore d is a metric on $\mathbb{R}^2$. ■

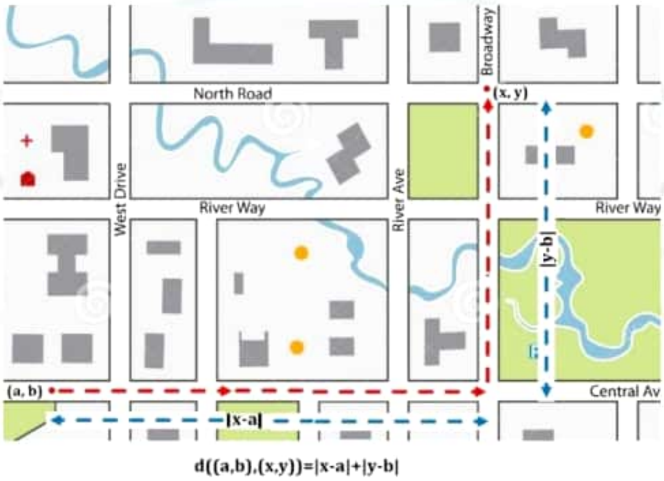


Figure 2. Road Map of a City

Example 4.5 Maximum Metric on $\mathbb{R}^2$

Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers and $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined as

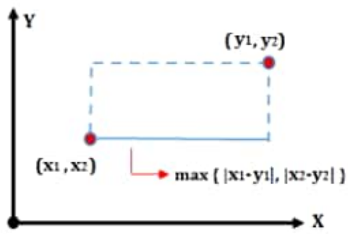
$$d(x, y) = \max \{|x_1 - y_1|, |x_2 - y_2|\} \qquad \forall\ x = (x_1, x_2),\ \ y = (y_1, y_2) \in \mathbb{R}^2.$$

We shall show that d is a metric on $\mathbb{R}^2$.

By definition, d is a non-negative function and hence **P1** holds.

For **P2**, consider any $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$,

$$\begin{aligned} d(x, y) = 0 &\Leftrightarrow \max \{|x_1 - y_1|, |x_2 - y_2|\} = 0 \\ &\Leftrightarrow |x_1 - y_1| = 0 \ \ and \ \ |x_2 - y_2| = 0 \\ &\Leftrightarrow x_1 = y_1 \ \ and \ \ x_2 = y_2 \\ &\Leftrightarrow (x_1, x_2) = (y_1, y_2) \ \ i.e., \quad x = y \ . \end{aligned}$$



For any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$,

$$\begin{aligned} d(x, y) &= \max \{|x_1 - y_1|, |x_2 - y_2|\} \\ &= \max \{|y_1 - x_1|, |y_2 - x_2|\} \\ &= d(y, x)\,. \end{aligned}$$

Thus **P3** is satisfied.

To see triangle inequality **(P4),** let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$ be any points in $\mathbb{R}^3$. Consider

$$\begin{aligned} |x_1 - y_1| &= |(x_1 - z_1) + (z_1 - y_1)| \\ &\le |x_1 - z_1| + |z_1 - y_1| \end{aligned}$$

$$\leq \max\{|x_1 - z_1|, |x_2 - z_2|\} + \max\{|z_1 - y_1|, |z_2 - y_2|\}$$
$$= d(x,z) + d(z,y)$$
$$i.e., \quad |x_1 - y_1| \leq d(x,z) + d(z,y) \qquad -------- (\boldsymbol{A})$$

Similarly,

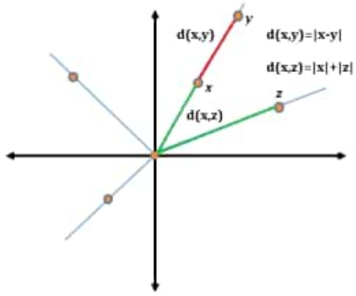
$$|x_2 - y_2| \leq d(x,z) + d(z,y)\ . \qquad -------- (\boldsymbol{B})$$

From (A) and (B) it follows that

$$\max\{|x_1 - y_1|, |x_2 - y_2|\} \leq d(x,z) + d(z,y)$$
$$i.e., \qquad d(x,y) \leq d(x,z) + d(z,y)\,.$$

Hence the triangle inequality holds and therefore d is a metric on $\mathbb{R}^2$. ■

Example 4.6 Let $X = \mathbb{R}^2$ and $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by



$$d(x,y) = \begin{cases} |x-y| & \textit{if } x \textit{ and } y \textit{ are in the same ray from the origin} \\ |x|+|y| & \textit{otherwise} \end{cases}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Show that d is a metric on $\mathbb{R}^2$. (Here $|x - y| = u(x,y)$ and $|x| = u(x,0)$ and u is Euclidean metric on $\mathbb{R}^2$.)

Proof: Clearly, $d(x,y) \geq 0 \ \ \forall\, x, y \in \mathbb{R}^2$.

For any $x, y \in \mathbb{R}^2$

$$d(x,y) = \begin{cases} |x-y| & \textit{if } x \textit{ and } y \textit{ are in the same ray from the origin} \\ |x|+|y| & \textit{otherwise} \end{cases}$$
$$= \begin{cases} u(x,y) & \textit{if } x \textit{ and } y \textit{ are in the same ray from the origin} \\ u(x,0)+u(y,0) & \textit{otherwise} \end{cases}$$
$$= \begin{cases} u(y,x) & \textit{if } x \textit{ and } y \textit{ are in the same ray from the origin} \\ u(y,0)+u(x,0) & \textit{otherwise} \end{cases}$$
$$= \begin{cases} |y-x| & \textit{if } x \textit{ and } y \textit{ are in the same ray from the origin} \\ |y|+|x| & \textit{otherwise} \end{cases}$$
$$= d(y,x)\,.$$

By definition of d, observe that

$$d(x,y) \geq |x-y| \quad \forall\ x, y \in \mathbb{R}^2\,. \qquad -------- (A) \ \ [\because |x-y| \leq |x| + |y|]$$

Thus for any $x, y \in \mathbb{R}^2$

$$d(x,y) = 0 \ \Rightarrow \ |x-y| = 0 \ \Rightarrow x = y\,.$$

Also, $x = y$ implies that x and y are in the same ray from the origin and therefore

$$d(x,y) = |x-y| = 0\,.$$

Finally to prove triangle inequality, consider any $x, y, z \in \mathbb{R}^2$.

<u>**Case I**</u> x and y are in the same ray from the origin

Then

$$d(x,y) = |x-y|$$
$$= u(x,y)$$
$$\leq u(x,z) + u(z,y)$$
$$= |x-z| + |z-y|$$
$$\leq d(x,z) + d(z,y) \qquad [Using\ \ (A)]$$

Case II x and y are in the different ray from the origin.

Subcase I z and x are in different ray from the origin

Then

$$\begin{aligned} d(x,y) &= |x| + |y| \\ &= u(x,0) + u(0,y) \\ &\le u(x,0) + [u(0,z) + u(z,y)] \\ &= |x| + [|z| + |y-z|] \\ &= [|x| + |z|] + |y-z| \\ &\le d(x,z) + d(y,z) \qquad [\because |y-z| \le d(y,z) \ldots.(A)] \\ &= d(x,z) + d(z,y) \end{aligned}$$

Subcase II z and x are in same ray from the origin.

Then z and y are in different ray from the origin. Therefore

$$\begin{aligned} d(x,y) &= |x| + |y| \\ &= u(x,0) + u(0,y) \\ &\le u(x,z) + u(z,0) + u(0,y) \\ &= |x-z| + |z| + |y| \\ &= d(x,z) + d(z,y). \end{aligned}$$

Thus in all the cases triangle inequality is satisfied and hence d is a metric on $\mathbb{R}^2$. ■

Railway Metric

The metric given in Example 4 is called the ***Railway metric*** as it can be used to describe the following situation (hypothetical).

Consider a proposed metro network of India for 2030 where all the major towns lie on some metro track originating from Delhi (see adjoining Figure). Thus on this network, one can travel directly between any two towns which lie on the same metro track to Delhi. Otherwise first one has to go Delhi and change to another line.