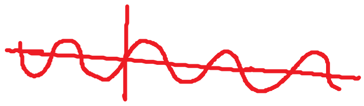
25th September 2020

Example 5.2.4 (c) $\sin x$ is cts on $\mathbb{R}$ $f(x) = \sin x$

→ combination of cts fns
→ polynomial fn
→ rational expression
$|x|$,



Let $c \in \mathbb{R}$ & $\varepsilon > 0$

Consider,

$$|f(x) - f(c)| = |\sin x - \sin c|$$

$$= \left|2 \sin\left(\frac{x-c}{2}\right) \cos\left(\frac{x+c}{2}\right)\right|$$

$$= 2 \left|\sin\left(\frac{x-c}{2}\right)\right| \left|\cos\left(\frac{x+c}{2}\right)\right| \qquad |a.b| = |a|.|b|$$

$$\leq 2 \left|\sin\left(\frac{x-c}{2}\right)\right| \quad \text{as } |\cos\theta| \leq 1$$

$$\leq \cancel{2} \left|\frac{x-c}{\cancel{2}}\right| \quad \text{as } |\sin\theta| \leq |\theta| \ \forall \theta \quad (1)$$

Let $\delta = \varepsilon$ ∴ from (1)

If $|x-c| < \delta$ then $|f(x) - f(c)| < \varepsilon$

i.e $\lim_{x \to c} \sin x = \sin c$

d) $\cos x$ is continuous on $\mathbb{R}$ (Ex)

e) $\sec x = \frac{1}{\cos x}$, $\cos x \neq 0$

$= \frac{f(x)}{g(x)}$, $g(x) \neq 0$

By algebra of cts fns as $f(x) = 1$ (constant functions)

& $g(x) = \cos x$ is cts on $\mathbb{R}$

∴ f/g is cts for $g \neq 0$

Illy $\operatorname{cosec} x$, $\tan x$, $\cot x$ are cts wherever they are defined

Thm 5.2.4 Let $f : A \to \mathbb{R}$; $A \subseteq \mathbb{R}$ & $|f| : A \to \mathbb{R}$

$|f|(x) = |f(x)|$

(1) If f is cts at a point $c \in A$, then $|f|$ is cts at c

(2) If f is cts on A then $|f|$'s cts on A

$f \to L$ at c ($L = f(c)$)

$|f| \to |L|$ at c ($|L| = |f(c)|$)

$||f| - |L|| \leq |f - L|$

$x \to c \Rightarrow f \to L$, $f \geq 0$ ($L = f(c)$)

then $\sqrt{f(x)} \to \sqrt{L}$ at c

Proof (1) We know that if $\lim_{x \to c} f(x) = f(c)$ (as f is cts at c)

$x_n \to c$

$\Rightarrow f(x_n) \to f(c)$

$\Rightarrow |f(x_n)| \to |f(c)|$

$y_n \to y$ $\Rightarrow |y_n| \to |y|$

$\Rightarrow \lim_{x \to c} |f|(x) = |f(c)|$

$= |f|(c)$ $h = |f|$

$\Rightarrow f(x_n) \to f(c)$ // $\}$ on $\Downarrow$ $\Rightarrow \lim_{x\to c} |f|(x) = |f(c)|$

$\Rightarrow |f(x_n)| \to |f(c)|$ ($|f_n| \to |y|$) $= |f|(c)$ $\quad \underline{h = |f|}$

$\Rightarrow |f|$ is cts at c

② Now, let f be cts on A

$\Rightarrow f$ is cts at each $c \in A$

$\Rightarrow |f|$ is cts at each $c \in A$ (by ①)

$\Rightarrow |f|$ is cts on A

Thm 5.2.5 $\quad f : A \to \mathbb{R}$, $f \geq 0$ on A

① If f is cts at $c \in A$ then $\sqrt{f}$ is cts at $c \in A$

② If f is cts on A then $\sqrt{f}$ cts on A

Pf $\to$ Exercise

Exercise 5.2

Q1) Determine the points of continuity of the following fns and state which theorems are used.

ⓐ $f(x) = \dfrac{x^2+2x+1}{x^2+1}$, $x \in \mathbb{R}$, $\qquad x^2 = -1 \times x \in \mathbb{R}$

$p(x) = x^2+2x+1$, $q(x) = x^2+1$ are cts fns as poly fns are cts (on $\mathbb{R}$)

Also, $q(x) \neq 0 \ \forall x \in \mathbb{R}$ $\therefore \dfrac{p(x)}{q(x)}$ is cts (Algebra of cts fns)

ⓑ $g(x) = \sqrt{x+\sqrt{x}}$; $x \geq 0$

$x \to \sqrt{x}$ cts on $\mathbb{R}$

$x \to x$ cts on $\mathbb{R}$

$\Rightarrow x \to x+\sqrt{x}$ is cts

$\Rightarrow x \to \sqrt{x+\sqrt{x}}$ is cts ($x+\sqrt{x} \geq 0$)

$\Rightarrow g$ is cts (on $\mathbb{R}^+ \cup \{0\}$)

$\underline{x \geq 0}$

$f_1(x) = \sqrt{x}$

$f_2(x) = x$ (identity fn)

f_1, f_2 are cts on $\mathbb{R}^+ \cup \{0\}$

$\Rightarrow f_1 + f_2$ is cts (algebra of cts) (Sum)

$\Rightarrow \sqrt{f_1+f_2}$ is cts $\quad$ 5.2.5

$\Rightarrow g$ is cts ($f_1+f_2 \geq 0$)

ⓒ $h(x) = \dfrac{\sqrt{1+|\sin x|}}{x}$, $x \neq 0$

$f_1(x) = 1$

$f_2(x) = \sin x$

$f_3(x) = x$

$x \to 1$ cts (const)

$x \to \sin x$ is cts $\quad$ (f cts $\Rightarrow |f|$ cts)

$\Downarrow$

$x \to |\sin x|$ cts

$x \to \sin x$ is cts

$\Downarrow$

$x \to |\sin x|$ cts $\quad$ (f cts $\Rightarrow |f|$ cts) $\quad$ $b_3(x) = x$

$\Rightarrow_{x \to} 1 + |\sin x|$ cts (sum of cts functions)

$\Rightarrow_{x \to} \sqrt{1 + |\sin x|}$ cts $\quad$ $(1 + |\sin x| > 0)$ $\left(f \text{ cts} \Rightarrow \sqrt{f} \text{ cts}, f \geq 0\right)$

$\Rightarrow_{x \to} \dfrac{\sqrt{1 + |\sin x|}}{x}$ cts $\quad$ ($\because$ x is cts (identity fn) & algebra of cts f^ns)

d) $\quad k(x) = \cos\sqrt{1 + x^2}$, $x \in \mathbb{R}$

Q2 Show that if $f : A \to \mathbb{R}$ is cts on $A \subseteq \mathbb{R}$ & if $n \in \mathbb{N}$ then f^n is cts on $\mathbb{R}$ $\qquad (f^n)(x) = (f(x))^n$

$f \cdot f \quad f^2 \cdot f \quad f^3 \cdot f \quad f^n$

Proof $\quad f$ is cts on A

$\Rightarrow f \cdot f$ i.e. f^2 is cts ($\because$ algebra of cts f^ns)

Let f^k be cts on A for some $k \in \mathbb{N}$ (ind.)

$\Rightarrow f^k \cdot f$ is cts on A (algebra of cts f^ns)

$f^k \cdot f$

i.e. f^{k+1} is cts on A

Hence, by induction $\forall n \in \mathbb{N}$, f^n is cts on A

Q3 Give an example of f^ns f and g that are both discts at $c \in \mathbb{R}$ but $\qquad f \to L$, $g \to M$

① sum $f+g$ is cts at c

② the product $f \cdot g$ is cts at c

Sol.

$f(x) = \operatorname{sgn} x \qquad g(x) = -\operatorname{sgn} x \qquad f(x) = \operatorname{sgn} x = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

$(f+g)(x) = 0 \quad \& \quad (f+g)(0) = 0$

$\Rightarrow \lim_{x \to 0} (f+g)(x) = 0 = (f+g)(0)$

$\therefore f+g$ is cts at 0 but f & g are discts at 0

Now, let $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ $\qquad g = -f$

Then $(fg)(x) = -1 \quad \forall x \qquad \operatorname{sgn} x = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ & x = 0 \end{cases}$

As $(fg)(x) = -1 \quad \forall x \qquad f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$\Rightarrow \lim_{x \to 0} (fg)(x) = -1$

Now $(fg)(0) = -1 \qquad g(x) = \left\{ \frac{x}{|x|}, \; x \neq 0 \right.$

$\Rightarrow \lim_{x\to 0}(fg)(x) = -1$

Also $(fg)(0) = -1$

$\therefore \lim_{x\to 0}(fg)(x) = (fg)(0)$

$\Rightarrow fg$ is cts at 0

However, f & g are discts at 0.

$g(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ -1 & x = 0 \end{cases}$

$\sqrt{x}$

$x \to \sqrt{x}$

Q4 Let $x \to [[x]]$ be the greatest integer function. Determine the points of cont of $x - [[x]]$

$[[\sin x]]$

Sol $x \to x$ is identity f^n, cts on $\mathbb{R}$

$x \to [[x]]$ is cts at $c \in \mathbb{R} \sim \mathbb{Z}$

$\therefore x \to x$ & $x \to [[x]]$ are cts at $c \in \mathbb{R} \sim \mathbb{Z}$ on $\mathbb{R} \sim \mathbb{Z}$

$[[x]]$ x

$\Rightarrow x \to x - [[x]]$ is cts on $\mathbb{R} \sim \mathbb{Z}$ (diff of cts fns)

Let $c \in \mathbb{Z}$ (we'll show $x \to x - [[x]]$ is discts at c)

Let $x_n = c - \frac{1}{n}$ $\forall n$ $(c-1 \leq x_n < c)$

$x_n \to c$, $f(x_n) = x_n - [[x_n]]$

$= c - \frac{1}{n} - (c-1)$,

$= 1 - \frac{1}{n}$

$\to 1$

$f(c) = c - [[c]]$

$= c - c$

$= 0$

$\therefore \exists\, x_n \to c : \lim f(x_n) \neq f(c)$

$\Rightarrow f$ is discts at c (by descont crit.)

Hence, f is discts on $\mathbb{Z}$ & cts on $\mathbb{R} \sim \mathbb{Z}$.

Q7 Give an example of a function which is discts at every point of $[0,1]$ but such that $|f|$ is cts on $[0,1]$

Sol Let $f(x) = \begin{cases} 1 & x \text{ is rational in } [0,1] \\ -1 & x \text{ is irrational in } [0,1] \end{cases}$ (f cts $\Rightarrow |f|$ cts. Is the conv. true?)

Then f is discts on $[0,1]$ $\xrightarrow{\text{Similar to}}$ Dirichlet's f^n $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \sim \mathbb{Q} \end{cases}$ a b

Now $|f|(x) = 1$ $\forall x \in [0,1]$

$\Rightarrow |f|$ is constant on $[0,1] \Rightarrow |f|$ is cts on $[0,1]$

Q8 Let f, g be cts from $\mathbb{R}$ to $\mathbb{R}$ $f(x) = g(x)$ $\forall x \in \mathbb{Q}$ $\to$ S.1 f cts

show that

Q8 Let f, g be cts from $\mathbb{R}$ to $\mathbb{R}$ $\quad f(x) = g(x)\ \forall x \in \mathbb{Q}$

Then show that $f(x) = g(x),\ \forall x \in \mathbb{R}$

Eg 5.1 f cts

$f(x) = 0\ \forall x \in \mathbb{Q}$

$\Rightarrow f(x) = 0\ \forall x \in \mathbb{R}$

©

Sol

$h = f - g$ on $\mathbb{R}$

h is cts on $\mathbb{R}$ (algebra of cts fs)

$x \in \mathbb{Q}$, $h(x) = (f-g)(x)$

$= f(x) - g(x)$

$= 0$

$\Rightarrow h(x) = 0\ \forall x \in \mathbb{R}$

$\Rightarrow (f-g)(x) = 0\ \forall x \in \mathbb{R} \Rightarrow f(x) = g(x)\ \forall x \in \mathbb{R}$

Composition of function

$\sin x^2$

$f(x) = x^2$

$g(x) = \sin x$

$g \circ f$

$f \circ g = (\sin x)^2$

$B \to \mathbb{R}$ such that

$B \subseteq \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R}$

$f(x)$ $\quad g(f(x))$

$g(f(x))$

$x \in A$, $f(x) \in B$

$f(A) \subseteq B$

$f : A \to \mathbb{R}$

$g : B \to \mathbb{R}$

$x \in A$, $g(f(x))$ $\quad A$ $\quad B$ $\quad \forall$

$f : \mathbb{R} \to \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R}$

$(g \circ f)(x)$

$\parallel$

$g(f(x))$ $\quad \mathbb{R}$

$f \circ g$ $\qquad g(B) \subseteq A$ $\qquad g(B) \subseteq A$

$y \in B$ $\qquad f(g(y))$ $\quad g(y) \in A$

Def: Let $f : A \to \mathbb{R}$ & $g : B \to \mathbb{R}$; $A, B \subseteq \mathbb{R}$

- $f(A) \subseteq B$ then

$g \circ f : A \to \mathbb{R}$ is defined as

$(g \circ f)(x) = g(f(x))\ \forall x \in A$

$g \circ f : A \longrightarrow \mathbb{R}$ is defined as

$$(g \circ f)(x) = g\big(f(x)\big) \quad \forall\, x \in A$$

Ex Define $f \circ g$.