Partial differential equations

Canonical forms (Laplace Transformation)

Consider the equation of the type

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0 \quad \text{——} \ (1)$$

where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary. Laplace transformation of (1) consists of changing the independent variables x, y to new set of independent variables u, v where

$$u = u(x,y) \text{ and } v = v(x,y) \quad \text{——} \ (2)$$

are to be chosen so that the resulting equation in independent variables u, v is transformed into one of three canonical forms, which are easily integrable.

From (2), we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}, \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \quad \text{——} \ (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y\partial x}$$

Example 8
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Reduce the equation
$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$ to canonical form

Soln

Given eqn can be written
$$r - x^2 t = 0 \quad \text{——— (1)}$$

Comparing ~~of~~ (1) with ~~the~~ $Rr + Ss + Tt + f(x, y, z, p, q) = 0$
we have
$$R = 1, \; S = 0, \; T = -x^2$$

Now the λ-quadratic ~~gives~~ $R\lambda^2 + S\lambda + T = 0$ gives
$$\cancel{R\lambda^2 +} \quad \lambda^2 - x^2 = 0 \Rightarrow \lambda = \pm x$$

$\therefore$ Here $\lambda_1 = x, \; \lambda_2 = -x$. (Real & distinct)

Hence $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes.

~~Integrating~~, ~~get~~ $\frac{dy}{dx} + x = 0$ and $\frac{dy}{dx} - x = 0$

Integrating, $y + \frac{x^2}{2} = c_1$ and $y - \frac{x^2}{2} = c_2$

Hence in order to reduce (1) to canonical form, we change x, y to u, v by taking
$$u = y + \frac{1}{2}x^2, \quad v = y - \frac{1}{2}x^2 \quad \text{——— (2)}$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x \cdot \frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$
$$= x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 1 \cdot \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= x\left[\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v}\right)\cdot x + \left(\frac{\partial^2 z}{\partial v \partial u} - \frac{\partial^2 z}{\partial v^2}\right)(-x)\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= x^2\left[\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

and $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right)$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot 1 + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Putting these values of r and t in (1)

$$x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} - x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $\dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4x^2}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$

or $$\boxed{\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)}$$

which is the required canonical form of the given equation.

Example 9

Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

to canonical form and hence solve it.

Sol^n

Given that $r + 2s + t = 0$ — ①

Comparing ① with the equation $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

$R = 1,\ S = 2,\ T = 1$

Now the λ-quadratic $R\lambda^2 + S\lambda + T = 0$ gives.

$$\lambda^2 + 2\lambda + 1 = 0$$

$$\Rightarrow \quad \lambda = -1, -1.$$

$\therefore$ Here $\lambda_1 = -1,\ \lambda_2 = -1$

Hence $\frac{dy}{dx} + \lambda_1 = 0$ becomes

$$\frac{dy}{dx} - 1 = 0 \Rightarrow y = x + c_1$$

Hence in order to reduce ① to canonical form, we change x, y to u, v by taking

$u = x - y$ and $v = x + y$ — ②

$$\therefore\ p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$= -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Putting these values of r, s, t in ①.

$$\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + 2\left(-\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

$$+ \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = 0$$

$$\Rightarrow 4\frac{\partial^2 z}{\partial v^2} = 0 \quad \Rightarrow \frac{\partial^2 z}{\partial v^2} = 0$$

Integrating, w.r.t 'v', $\frac{\partial z}{\partial v} = \phi_1(u)$

Again integrating,

$$z = v\phi_1(u) + \phi_2(u)$$

$$\Rightarrow \boxed{z = (x+y)\phi_1(x-y) + \phi_2(x-y)}$$

Example 10 Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{to canonical form.}$$

Sol^n Given eq^n can be written as

$$r + x^2 t = 0 \quad \text{——} (1)$$

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

$$R = 1, \; S = 0, \; T = x^2$$

Now the λ-quadratic $\lambda^2 R + \lambda S + T = 0$ gives

$$\lambda^2 + x^2 = 0 \Rightarrow \lambda = \pm ix$$

$\therefore$ here $\lambda_1 = ix, \; \lambda_2 = -ix.$

Hence $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes

$$\frac{dy}{dx} + ix = 0 \quad \text{and} \quad \frac{dy}{dx} - ix = 0$$

$$\Rightarrow \quad y + i\frac{x^2}{2} = c_1 \quad \text{and} \quad y - i\frac{x^2}{2} = c_2$$

Hence in order to reduce (1) to canonical form, we change x, y to u, v by taking

$$u = y + \frac{ix^2}{2}, \quad v = y - \frac{ix^2}{2}$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot(ix) + \frac{\partial z}{\partial v}\cdot(-ix)$$

$$= ix\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[ix\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$$

$$= i\left[x\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 1\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$$

$$= ix\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] + i\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= ix\left[\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v}\right)(ix) + \left(\frac{\partial^2 z}{\partial v \partial u} - \frac{\partial^2 z}{\partial v^2}\right)(-ix)\right] + i\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= x^2\left[-\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} - \frac{\partial^2 z}{\partial v^2}\right] + i\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \cdot \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Putting these values of r and t in ①

$$x^2\left[-\cancel{\frac{\partial^2 z}{\partial u^2}} + 2\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2}\right] + i\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$
$$+ x^2\left(\cancel{\frac{\partial^2 z}{\partial u^2}} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

$$\Rightarrow \quad 4x^2\frac{\partial^2 z}{\partial u \partial v} = -i\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial u \cdot \partial v} = \frac{-i}{4x^2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial u \cdot \partial v} = -\frac{i^2}{u-v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial u \cdot \partial v} = \frac{1}{u-v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

Qu 4
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Reduce the equation
$$(x-1)^2\frac{\partial^2 z}{\partial x^2} - y^{2n}\frac{\partial^2 z}{\partial y^2} = ny^{2n-1}\frac{\partial z}{\partial y}$$
to canonical form, and hence its general solutions.

Solution Given that $(x-1)^2 r - y^{2n} t - ny^{2n-1} q = 0$ — (1)

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$
$$R = (x-1)^2, \quad S = 0, \quad T = -y^{2n}.$$

Now, the λ-quadratic $R\lambda^2 + S\lambda + T = 0$ gives
$$(x-1)^2\lambda^2 - y^{2n} = 0 \Rightarrow \lambda = \pm (x-1)^{-1}y^n.$$

$\therefore$ here $\lambda_1 = (x-1)^{-1}y$ and $y_2 = -(x-1)^{-1}y^n$.

Hence $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes.
$$\frac{dy}{dx} + (x-1)^{-1}y^n = 0 \quad \text{and} \quad \frac{dy}{dx} - (x-1)^{-1}y^n = 0$$

Integrating, $x - y^{-n+1} = c_1$ and $x + y^{-n+1} = c_2$

Hence in order to reduce (1) to canonical form, we change x, y to u, v by taking
$$u = x - y^{-n+1} \quad \text{and} \quad v = x + y^{-n+1}$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = (x-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$
$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{(n-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)^2 y^{-2n}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

Putting these values in ①, we obtain

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{③}$$

which is the required canonical form of the given equation.

Integrating w.r.t. 'v', $\frac{\partial z}{\partial u} = \phi(u)$

where $\phi(u)$ is an arbitrary f^n of u.

Integrating, $z = \psi(u) + \xi(v)$

where $\psi(u) = \int \phi(u)\,du$ and $\xi(v)$ are arbitrary f^{ns}.

using ②,

$$\boxed{z = G(x - y^{-n+1}) + H(x + y^{-n+1})}$$

Ques
p-109

Reduce the equation $y^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + x^2\frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x}\frac{\partial z}{\partial x} + \frac{x^2}{y}\frac{\partial z}{\partial y}$

to canonical form, and hence solve it.

Solⁿ

Given that $y^2 r - 2xys + x^2 t = \frac{y^2}{x}p + \frac{x^2}{y}q$. — (1)

here $R = y^2$, $S = -2xy$, $T = x^2$

$\therefore$ λ-quadratic gives $R\lambda^2 + S\lambda + T = 0$

$y^2\lambda^2 - 2xy\lambda + x^2 = 0$

$\Rightarrow (y\lambda - x)^2 = 0 \Rightarrow \lambda = \frac{x}{y}$

Hence $\frac{dy}{dx} + \lambda = 0$ becomes

$\frac{dy}{dx} + \frac{x}{y} = 0 \Rightarrow x^2 + y^2 = c_1$

$$\left.\begin{aligned} \therefore\ u &= x^2 + y^2 \\ \text{assume,}\quad v &= x^2 - y^2 \end{aligned}\right\} \quad \text{—(2)}$$

$\therefore\ p = 2x\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}$

$q = 2y\frac{\partial z}{\partial u} - 2y\frac{\partial z}{\partial v}$

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right) = 2\frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\}$

$= 2\left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + 1.\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right]$

$= 2x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] + 2\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$

$= 4x^2\left[\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right] + 2\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$

Similarly, we can find

$$s = 4xy\frac{\partial^2 z}{\partial u^2} - 4xy\frac{\partial^2 z}{\partial v^2}$$

and $$t = 4y^2\frac{\partial^2 z}{\partial u^2} - 8y^2\frac{\partial^2 z}{\partial u \partial v} + 4y^2\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial u} - 2\frac{\partial z}{\partial v}$$

Putting these values in ①, we get.

$$\frac{\partial^2 z}{\partial v^2} = 0 \quad \text{—— ③}$$

which is the required canonical form of the given eqn.

Integrating ③, $\frac{\partial z}{\partial v} = \phi_1(u)$

$$z = v\,\phi_1(u) + \phi_2(u)$$

$$\boxed{z = (x^2-y^2)\,\phi_1(x^2+y^2) + \phi_2(x^2+y^2)}$$