Example: Every chain is a distributive lattice. (Verify)

example: Every factorization lattice is distributive.

$\therefore$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

& $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

i.e. $\gcd\{x, \mathrm{lcm}(y,z)\} = \mathrm{lcm}\{\gcd(x,y), \gcd(x,z)\}$

& $\mathrm{lcm}\{x, \gcd\{y,z\}\} = \gcd\{\mathrm{lcm}\{x,y\}, \mathrm{lcm}\{x,z\}\}$

Verify

example If A = any set.

$P(A)$ is a distributive lattice.

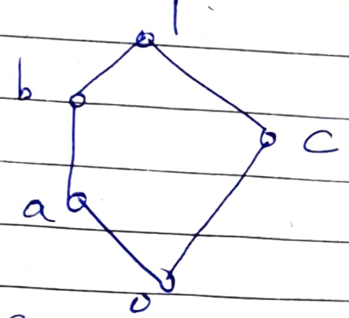
$\therefore$ $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

& $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

Remark Any lattice with 4 elements or less is distributive.

Pentagonal lattices & Partition lattices are smallest non distributive lattices.

Pentagonal lattice



$b \wedge (a \vee c) = b \wedge 1 = b$.

$(b \wedge a) \vee (b \wedge c) = a \vee 0 = a$

$\therefore$ Pentagonal lattice is non distributive.

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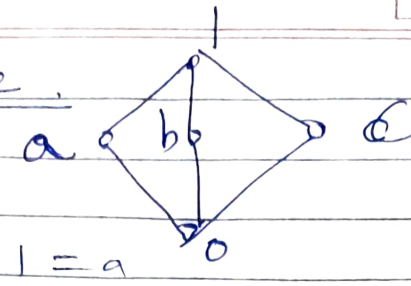
Partition lattice.



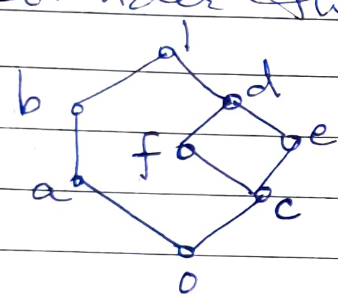
$a \wedge (b \vee c) = a \wedge 1 = a$

$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

$\therefore$ Partition lattice is non distributive

Theorem : A lattice is distributive iff it does not contain a sublattice isomorphic to a diamond (Partition) or Pentagonal lattice.

ex Consider the lattice.



L is a lattice

$S = \{0, 1, a, b, e\}$ is a sublattice of L.

& $S \cong$ Pentagonal lattice

$\therefore$ L is non distributive

Thm A lattice ~~is~~ L is distributive iff the cancellation rule

$$\left.\begin{array}{l} x \wedge y = x \wedge z \\ x \vee y = x \vee z \end{array}\right] \Rightarrow \cancel{x}\; y = z$$

holds $\forall x, y, z \in L$.

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Pf let L be a distributive lattice
let $x \wedge y = x \wedge z$ & $x \vee y = x \vee z$

Then $y = y \wedge (y \vee x)$ (By absorption)
$= y \wedge (x \vee z)$
$= (y \wedge x) \vee (y \wedge z)$
$= (x \wedge z) \vee (y \wedge z)$
$= (x \vee y) \wedge z$
$= (x \vee z) \wedge z = z$

$\therefore y = z$.

Defn Complemented lattice -

A lattice L with 0 & 1 is called Complemented if for each $x \in L$, $\exists\, y \in L$ s.t $x \wedge y = 0$ & $x \vee y = 1$.

ex $A = \{1,2,3\}$

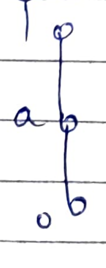
$P(A)$ is a lattice with zero $= \emptyset$ & Unit element $= A$.

If $X \in P(A)$, $\exists\, Y = A - X \in P(A)$ s.t $X \cap Y = \emptyset$ & $X \cup Y = A$

$\therefore$ $P(A)$ is a Complemented lattice.

Remark: In any bdd. lattice L,
Comp. of 0 is 1
& Comp. of 1 is 0.

~~Rema~~ **ex** Chain of 3 elements is not Complemented.



Complement of a does not exist.

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Any chain with 3 or more elements is not Complemented.

ex $(\mathbb{N}, \leq)$ is not Complemented.

ex $(\mathbb{N}, \text{'|'})$ is not Complemented.

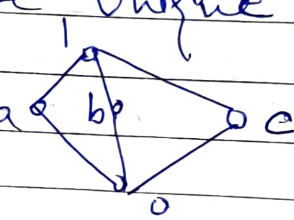
ex L = factors of 54.

L is a lattice w.r.t. $\wedge = \gcd$ & $\vee = \text{lcm}$.

L is a Complemented lattice.

Remark: The Complement of an element may ~~or m~~ not be Unique.

ex Partition lattice



$a \wedge b = 0, \quad a \vee b = 1$

Comp. of a is b

$a \wedge c = 0, \quad a \vee c = 1$

Comp. of a is c also.

Result If L is a distributive lattice, then each $x \in L$ has at most one Complement.

Pf Let L be a distributive lattice. Let y & y' are 2 Complements of x in L.

Then
$$
\begin{aligned}
y' &= y' \wedge (y' \vee x) \\
&= y' \wedge (y \vee x) \\
&= (y' \wedge y) \vee (y' \wedge x) \\
&= (y' \wedge y) \vee (y \wedge x) = y \wedge (y' \vee x) \\
&= y \wedge (y \vee x) \\
&= y.
\end{aligned}
$$

$\therefore$ Comp. of x (if it exists) is Unique.