Complex Analysis

Problems based on Cauchy Integral Formula

Q1. $\int_C \frac{z}{z^2+9}\,dz$ where C is the circle $|z-2i|=4$.

By Factorizing the denominator

$z^2+9 = (z+3i)(z-3i)$

we note that $z=3i$ is the only point within the closed contour C at which the integrand fails to be analytic.

Then by re-writing the integrand as

$$\frac{z}{z^2+9} = \frac{\left.\frac{z}{z+3i}\right\} f(z)}{z-3i}$$

we can identify $f(z) = z/(z+3i)$

The function f is analytic at all points within and on the contour C.

Hence, from Cauchy's Integral formula, we have

$$\int_C \frac{z}{z^2+9}\,dz = \int_C \frac{\frac{z}{(z+3i)}}{(z-3i)}\,dz = 2\pi i\, f(3i)$$

$$= 2\pi i \cdot \left(\frac{3i}{6i}\right)$$

$$= \pi i.$$

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Q2.

$$\int_C \frac{z+1}{z^4+2iz^3}\,dz \quad \text{where } C: |z|=1.$$

Inspection of the integrand shows that it is not analytic at $z=0$ and $z=-2i$ but only $z=0$ lies within the closed contour.

By writing the integrand as.

$$\frac{z+1}{z^4+2iz^3} = \frac{\frac{z+1}{z+2i}}{z^3} = \frac{\frac{z+1}{z+2i}}{(z-0)^3}$$

we can identify, $z_0=0$, $n=2$ and $f(z)=\frac{z+1}{z+2i}$

Then by Cauchy Integral Formula for Derivatives

$$\int_C \frac{z+1}{z^4+2iz^3}\,dz$$

$$= \int_C \frac{f(z)}{(z-0)^3}\,dz = \frac{2\pi i}{2!} f''(0) = -\frac{\pi}{4}+\frac{\pi}{2}i$$

$$f''(z) = \frac{2-4i}{(z+2i)^3} \quad \text{(By Quotient Rule)}$$

$$\therefore \; f''(0) = \frac{2i-1}{4i}$$

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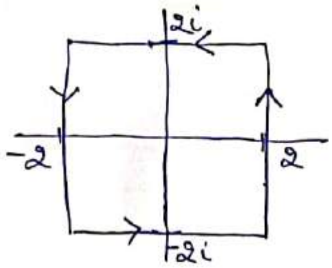
Exercise (Page 170-171)

Q1 Let C denote the +vely oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of the following integrals:

(a) $\int_C \frac{e^{-z} dz}{z - (\pi i/2)}$ (b) $\int_C \frac{\cos z}{z(z^2+8)}$ (c) $\int_C \frac{z\,dz}{2z+1}$

(d) $\int_C \frac{\cosh z}{z^4} dz$ (e) $\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz \quad (-2 < x_0 < 2)$.

Soln. The contour is represented as follows:

C: 

(a) Note that $z_0 = \frac{\pi i}{2}$ lie inside C

and $f(z) = e^{-z}$ is analytic inside and on C

$\therefore$ By Cauchy Integral Formula

$$\int_C \frac{e^{-z}}{(z - \pi i/2)} dz = 2\pi i\, f(z_0) = 2\pi i\, e^{-\frac{\pi i}{2}}$$
$$= 2\pi i\,(\cos(-\pi/2) + i\sin(-\pi/2))$$
$$= 2\pi i\,(0 - i)$$
$$= -2\pi i^2$$
$$= -2\pi(-1) = 2\pi$$

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(b)

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz$$

Note that $z=0$ is the only point in C at which integrand fails to be analytic.

Re-writing integrand as

$$\frac{\cos z}{z(z^2+8)} = \frac{\frac{\cos z}{(z^2+8)}}{z} = \frac{f(z)}{(z-0)}$$

The function $f(z) = \frac{\cos z}{z^2+8}$ is analytic inside and on the contour C (Note: $z^2+8 = (z-2\sqrt{2}i)(z+2\sqrt{2}i)$, $2\sqrt{2}i$ & $-2\sqrt{2}i$ both lie outside the C)

$\therefore$ By Cauchy Integral Formula

$$\int_C \frac{\cos z}{(z^2+8)z}\,dz = \int \frac{f(z)}{(z-0)}\,dz = 2\pi i\, f(0)$$

$$= 2\pi i \times \frac{1}{8} = \pi i/4.$$

(c) Do yourself.

(d)

$$\int_C \frac{\cosh z}{z^4}\,dz$$

$f(z) = \cosh z$ is analytic inside and on C

$\therefore$ By Cauchy Integral formula for derivatives

$$\int_C \frac{\cosh z}{z^4}\,dz = \int_C \frac{f(z)}{(z-0)^4} = \frac{2\pi i}{3!}\, f'''(0)$$

$$= \frac{2\pi i}{6} \times 0 = 0$$

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(e) $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz \qquad (-2 < x_0 < 2)$

$f(z) = \tan(z/2)$ is analytic inside and on C.

$\therefore$ By Cauchy Integral formula for derivatives

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz = \frac{2\pi i}{1!} f'(x_0)$$

$$= 2\pi i \cdot \sec^2(x_0/2)\cdot\frac{1}{2}$$

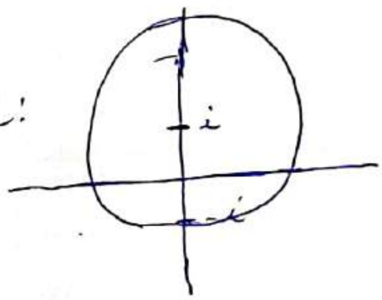
$$= \pi i \sec^2(x_0/2).$$

Q2. Find the value of integral of $g(z)$ around the circle $|z-i|=2$ in +ve sense, when

(a) $g(z) = \dfrac{1}{z^2+4}$ (b) $g(z) = \dfrac{1}{(z^2+4)^2}$

(a) Already done.

C:



(b) $\displaystyle\int_C \frac{1}{(z^2+4)^2}\,dz$

$$= \int_C \frac{1}{(z+2i)^2(z-2i)^2}\,dz$$

$z_0 = 2i$ lies inside C

and $f(z) = \dfrac{1}{(z+2i)^2}$ is analytic inside and on C

$\therefore$ By Cauchy Integral Formula for derivatives

$$\int_C \frac{1}{(z^2+4)^2}\,dz = \int_C \frac{f(z)}{(z-2i)^2}\,dz = \frac{2\pi i}{1!} f'(2i)$$

$$= \pi/16.$$

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Q3. $C: |z| = 3$, $g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds \quad (|z| \neq 3)$

(i) Show that $g(2) = 8\pi i$

(ii) What is the value of $g(z)$ when $|z| > 3$?

Soln.

$$g(2) = \int_C \frac{2s^2 - s - 2}{s - 2}\, ds$$

Let $f(s) = 2s^2 - s - 2$ and $s_0 = 2$

As f is analytic inside and on C

and $s_0 = 2$ is interior to C

$\therefore$ By Cauchy Integral Formula

$$\int_C \frac{2s^2 - s - 2}{s - 2}\, ds = 2\pi i\, f(2) = 2\pi i (4) = 8\pi i$$

$$\therefore \boxed{g(2) = 8\pi i}$$

(ii) Let $h(s) = \dfrac{2s^2 - s - 2}{s - z}$

~~When $|z| > 3$~~

h is analytic inside and on C when $|z| > 3$.

$\therefore$ By Cauchy-Goursat Theorem (Note that C is a simple closed contour)

$$\int_C h(s)\, ds = 0$$

$\therefore$ $g(z) = 0$ when $|z| > 3$.

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Q4. Let C be any simple closed contour, described in the +ve sense in the z-plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s-z)^3}\, ds$$

(i) Show that $g(z) = 6\pi i z$ when z is inside C

(ii) Show that $g(z) = 0$ when z is outside C

Hint: Do as per Question 3.

Q5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\, dz}{z - z_0} = \int_C \frac{f(z)\, dz}{(z - z_0)^2}$$

Soln. As f is analytic within and on a simple closed contour C

Then by Cauchy Integral formula for derivatives

$$\int_C \frac{f(z)}{(z-z_0)^2}\, dz = 2\pi i\, f'(z_0) \quad \text{--- (1)}$$

Since f is analytic within and on C

$\therefore$ all its derivatives are also analytic within and on C,

$\therefore$ f' is analytic within and on C.

$\therefore$ Applying Cauchy Integral formula for f' at $z = z_0$ we have

$$f'(z_0) = \frac{1}{2\pi i} \int \frac{f'(z)}{(z - z_0)}\, dz \quad \text{--- (2)}$$

Or $$\int_C \frac{f'(z)}{z-z_0}\,dz = 2\pi i\, f'(z_0) \quad -(3)$$

From (1) & (3), we get

$$\int_C \frac{f'(z)}{z-z_0}\,dz = \int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

Q7. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$.

First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$$

Soln. Since $f(z) = e^{az}$ is entire and $z_0 = 0$ lie inside C

$\therefore$ By Cauchy Integral Formula

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i\, f(0) = 2\pi i \times 1 = 2\pi i \quad -(1)$$

Subsitute $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$ in (1), we get

$$\int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}} \cdot i \cdot e^{i\theta}\,d\theta = 2\pi i$$

$$\Rightarrow \quad i\int_{-\pi}^{\pi} e^{ae^{i\theta}}\,d\theta = 2\pi i$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} e^{ae^{i\theta}}\,d\theta = 2\pi$$

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$$\Rightarrow \int_{-\pi}^{\pi} e^{a(\cos\theta + i\sin\theta)}\, d\theta = 2\pi$$

$$\Rightarrow \int_{-\pi}^{\pi} e^{a\cos\theta} \cdot e^{ia\sin\theta}\, d\theta = 2\pi$$

$$\Rightarrow \int_{-\pi}^{\pi} e^{a\cos\theta} \left[\cos(a\sin\theta) + i\sin(a\sin\theta)\right] d\theta = 2\pi$$

Comparing Both sides, we get

$$\int_{-\pi}^{\pi} e^{a\cos\theta} \cos(a\sin\theta)\, d\theta = 2\pi$$

Since $\cos\theta$ is an even function

$$\therefore\ 2\int_{0}^{\pi} e^{a\cos\theta} \cos(a\sin\theta) = 2\pi$$

$$\Rightarrow \int_{0}^{\pi} e^{a\cos\theta} \cos(a\sin\theta) = \pi.$$

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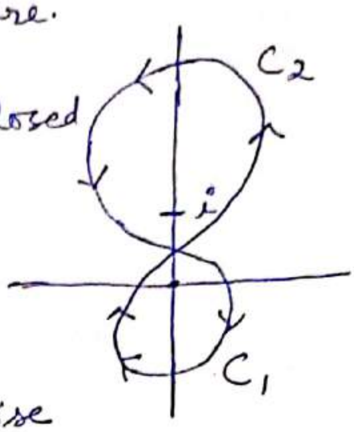
Extra Question.

Evaluate $\int_C \frac{z^3+3}{z(z-i)^2}\,dz$, where C is the figure-eight contour shown in the figure.

Soln. Although C is not a simple closed contour, we can think of it as a union of two simple closed contours C_1 & C_2.



Since the arrows on C_1 flow clockwise or in the negative direction, the opposite curve $-C_1$ has +ve orientation. Hence we write

$$\int_C \frac{z^2+3}{z(z-i)^2}\,dz = \int_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \int_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz$$

$$= -\int_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}\,dz + \int_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}\,dz$$

$$= -I_1 + I_2.$$

To evaluate I_1,

we identify $z_0 = 0$, $f(z) = \frac{z^3+3}{(z-i)^2}$ and $f(0) = -3$

$\therefore$ By Cauchy Integral Formula,

$$I_1 = \int_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}\,dz = 2\pi i\, f(0) = 2\pi i(-3)$$
$$= -6\pi i$$

To evaluate I_2,

we now identify $z_0 = i$, $n = 1$, $f(z) = \frac{z^3+3}{z}$

$$f'(z) = \frac{2z^3-3}{z^2}$$

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$\therefore$ By Cauchy Integral formula for derivatives

$$I_2 = \int_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2} dz = \frac{2\pi i}{1!} f'(i) = 2\pi i\,(3+2i) = -4\pi + 6\pi i$$

$$\therefore \int_C \frac{z^3+3}{z(z-i)^2} dz = -I_1 + I_2 = 6\pi i + (-4\pi + 6\pi i)$$

$$= -4\pi + 12\pi i.$$

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Sec 52: Some Consequences of the Extension
Page 168

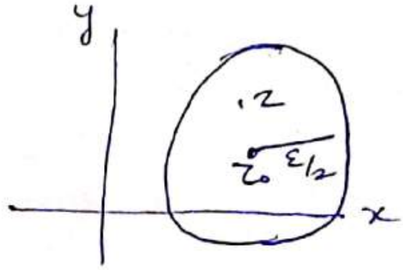
Theorem 1. If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.

Proof. Let f be analytic at a point z_0.

Then $\exists$ a nbhd $|z-z_0|<\varepsilon$ of z_0 throughout which f is analytic

Consequently, there is a +vely oriented circle C_0 centered at z_0 and radius $\varepsilon/_2$ such that f is analytic inside and on C_0

$\therefore$ By the extension of Cauchy Integral Formula,



$$f''(z) = \frac{2!}{2\pi i} \int_{C_0} \frac{f(s)}{(s-z)^3}\, ds$$

at each z interior to C_0

Now, the existence of $f''(z)$ throughout the nbhd $|z-z_0|<\varepsilon/_2$ implies that f' is analytic at z_0.

By applying the same argument to the analytic function f', we can conclude that f'' is analytic.

$\therefore$ The derivatives of all orders of f^n exist and are analytic at z_0.

Since z_0 was chosen arbitrarily, statement is true for any point.

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<u>Corollary</u>. If a function $f(z) = u(x,y) + i v(x,y)$ is analytic at a point $z = (x,y)$, then the component functions u and v have continuous partial derivatives of all orders at that point.

<u>Proof</u>. Since $f(z) = u(x,y) + i\, v(x,y)$ is analytic at a point $z = (x,y)$

$\therefore$ By previous theorem, its derivatives of all orders are also analytic there and hence are differentiable and therefore are continuous.

i.e. f' is continuous at $z = (x,y)$.

Since $f'(z) = u_x + i v_x = v_y - i u_y$

we conclude that first order partial derivatives of u and v are continuous at $z = (x,y)$.

Also, since f'' is analytic and hence continuous at z and since $f''(z) = u_{xx} + i v_{xx} = v_{yx} - i u_{yx}$

$\therefore$ Second order partial derivatives of u & v are also continuous.

Illy Partial derivatives of all orders of u & v are cts at $z = (x,y)$.

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Theorem 2. Let f be continuous on a domain D. If

$$\int_C f(z)dz = 0$$

for every closed contour C in D, then f is analytic throughout D.

Proof. As f is continuous on a domain D

and $\int_C f(z)dz = 0$ for every closed contour in D

$\therefore$ By Antiderivative theorem, f has an antiderivative in D

i.e. $\exists$ an analytic function F such that

$F'(z) = f(z)$ at each point z in D

Now, as F is analytic function in D,

By Theorem (1), its derivatives of all orders are analytic in D

$\therefore$ F' is analytic in D

$\Rightarrow$ f is analytic in D.

Theorem 3. Suppose that a function f is analytic inside and on a +vely oriented circle C_R, centered at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R, then

$$|f^n(z_0)| \leq \frac{n!\, M_R}{R^n} \quad (n = 1, 2, 3, \ldots)$$

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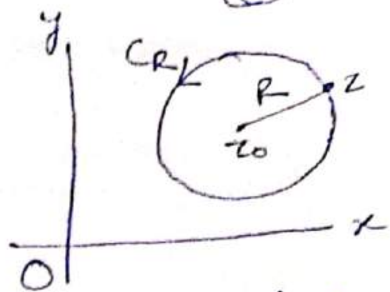
<u>Proof</u>. As f is analytic inside and on a +vely oriented circle C_R centered at z_0 and radius R



$\therefore$ By extension of Cauchy Integral Formula, we have

$$f^n(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad \text{—(1)}$$

Now, $|f(z)| \leq M_R \quad \forall z \in C_R$

By M-L Theorem,

$$|f^n(z_0)| = \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$$

$$\leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \times 2\pi R \qquad (\because |z-z_0| = R)$$

$$= \frac{n!\, M_R}{R^n} \qquad (n = 1, 2, 3, \ldots)$$

$$\therefore \boxed{|f^n(z_0)| \leq \frac{n!\, M_R}{R^n} \quad (n = 1, 2, 3 \ldots)}$$

This is called <u>Cauchy's Inequality</u>

Sec 53. Page 172 LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Liouville's Theorem Page 173: If a function is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Pf. As f is entire

$\therefore$ f is analytic at every $z_0 \in \mathbb{C}$

Let us fix z_0.

Then f is analytic inside and on a +vely oriented circle C_R centered at z_0 with radius R $(\forall R)$.

$\therefore$ By Cauchy's Inequality for $n=1$

$$|f'(z_0)| \le \frac{M_R}{R} \quad \text{--- (1)}$$

where M_R is the max. value of $|f(z)|$ on C_R

Since f is bdd on the complex plane,

$\therefore \exists\ M>0$ such that

$$|f(z)| \le M \quad \forall z \in \mathbb{C} \quad \text{--- (2)}$$

By (1) & (2),

$$|f'(z_0)| \le M/R \quad \text{--- (3)} \qquad (\text{Note that } M_R \le M)$$

where R can be arbitrarily large

Taking $R \to \infty$ in (3), we get

$$f'(z_0) = 0 \qquad (\text{Note: } M \text{ is independent of } R)$$

Since z_0 was chosen arbitrarily, $\therefore f'(z_0)=0 \ \forall z \in \mathbb{C}$

$\therefore$ f is constant on $\mathbb{C}$ (By Thm in Sec 24 on page 74)

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FUNDAMENTAL THEOREM OF ALGEBRA

Any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \neq 0)$$

of degree n $(n \geq 1)$ has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.

Pf. If possible, suppose that $\nexists$ any $z_0 \in \mathbb{C}$ s.t.

$$P(z_0) = 0$$

i.e. $P(z) \neq 0 \quad \forall z \in \mathbb{C}$

Define $f(z) = \dfrac{1}{P(z)}$, $z \in \mathbb{C}$.

Since every polynomial is an entire function

$\therefore$ f is also an entire function. —(*)

We claim that f is bounded on $\mathbb{C}$

Let us write

$$\omega = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \quad \text{—(1)}$$

Then $\omega = \dfrac{a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}}{z^n}$

$$\Rightarrow \omega = \frac{P(z) - a_n z^n}{z^n}$$

$$\Rightarrow P(z) = (a_n + \omega) z^n \quad \text{—(2)}$$

Now, for arbitrarily large $R > 0$

$$\left| \frac{a_{k-1}}{z^k} \right| \leq \frac{|a_{k-1}|}{R^k} \quad \text{when } |z| > 0$$

$k = 1, 2, 3, \ldots, n$

$$\longrightarrow 0 \quad \text{as } R \to \infty$$

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$\therefore$ We can make the modulus of each of the quotients in expression (1) less than the number $\dfrac{|a_n|}{2n}$ whenever $|z|>R$.

i.e. $$\left|\frac{a_{k-1}}{z^k}\right| < \frac{|a_n|}{2n} \quad \text{whenever } |z|>R$$
$k=1,2,3,\ldots,n$

$\therefore$ For $|z|>R$

$$|w| = \left|\frac{a_0}{z^n}+\frac{a_1}{z^{n-1}}+\cdots+\frac{a_{n-1}}{z}\right|$$

$$\leq \left|\frac{a_0}{z^n}\right|+\left|\frac{a_1}{z^{n-1}}\right|+\cdots+\left|\frac{a_{n-1}}{z}\right|$$

$$< \underbrace{\frac{|a_n|}{2n}+\frac{|a_n|}{2n}+\cdots+\frac{|a_n|}{2n}}_{(n \text{ times})}$$

$$= \frac{n|a_n|}{2n} = \frac{|a_n|}{2}$$

$$\therefore |a_n+w| \geq \big||a_n|-|w|\big|$$

$$> \left||a_n|-\frac{|a_n|}{2}\right|$$

$$= \frac{|a_n|}{2} \quad \text{whenever } |z|>R$$

$$\therefore |P_n(z)| = |a_n+w|\,|z|^n > \frac{|a_n|}{2}|z|^n$$

$$> \frac{|a_n|}{2}R^n \quad \text{whenever } |z|>R$$

$$\therefore |f(z)| = \left|\frac{1}{P(z)}\right| < \frac{2}{|a_n|R^n} \quad \text{whenever } |z|>R$$

$\therefore$ f is bounded in the region exterior to the ~~disk~~ disk $|z|\leq R$.

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i.e. f is bounded on the set $\{z \in \mathbb{C} : |z| > R\}$ — $(**)$

As, we have proved that $f(z)$ is entire (see $(*)$)

$\therefore$ It is continuous on the closed & bdd set $\{z \in \mathbb{C} : |z| \leq R\}$

$\therefore$ f is bdd on $\{z \in \mathbb{C} : |z| \leq R\}$ — $(***)$

By $(**)$ & $(***)$,

<u>f is bounded on $\mathbb{C}$</u> $= \{z \in \mathbb{C} : |z| > R\}$
$\cup \{z \in \mathbb{C} : |z| \leq R\}$

Thus we have proved that f is entire & bdd on $\mathbb{C}$

$\therefore$ By Liouville's Theorem, f is constant on the complex plane.

$\therefore$ $P(z)$ is constant on the complex plane

But $P(z)$ being a polynomial of degree $n \geq 1$ is a non constant polynomial

A contradiction !

$\therefore$ $\exists$ at least one $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

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Remark: A polynomial of degree $n (n \geq 1)$ can't have more than 'n' distinct roots.

The fundamental theorem of algebra tell us that any polynomial $P(z)$ of degree n $(n \geq 1)$ can be expressed as a product of linear factors.

$$P(z) = c(z-z_1)(z-z_2) \cdots (z-z_n)$$

where c & z_k $(k=1,2,3, \ldots n)$ are complex constants

Pf. As $P(z)$ is a polynomial of degree n $(n \geq 1)$

By Fundamental Thm of Algebra

$P(z)$ at least one zero (say z_1) in $\mathbb{C}$.

Let us write $P(z) = (z-z_1) Q_1(z)$, where $Q_1(z)$ is a poly. of degree $n-1$ $(n \geq 2)$

Now again applying FTOA to $Q_1(z)$

$Q_1(z)$ at least one zero (say z_2) in $\mathbb{C}$.

Then $P(z) = (z-z_1)(z-z_2) Q_2(z)$

where $Q_2(z)$ is a poly. of deg. $n-2$ $(n \geq 3)$

Continuing in this manner

All zeros of $P(z)$ lie in $\mathbb{C}$ (may be repeated more than once).

$\therefore P(z)$ can be written as

$$P(z) = c(z-z_1)(z-z_2) \cdots (z-z_k)$$

where $c, z_k \in \mathbb{C}$ $(k=1,2,\ldots,n)$

$\therefore P(z)$ can't have more than 'n' distinct roots.

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