

Permutation Group

①

A permutation of a set A is a function from A to A , which is both 1-1 and onto.

The set of all 1-1 onto mapping from set A to itself form a group w.r.t function composition is called permutation group.

If $A = \{1, 2, 3, \dots, n\}$ where n is some positive integer n , then grp form is called Symmetric grp, ^{denoted by} $\text{Sym}(n)$ or S_n . (Symmetric grp of degree n)

The elements of S_n are called permutations

A permutation $f \in S_n$ is denoted as

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ f(1) & f(2) & f(3) & \dots & f(n) \end{pmatrix}$$

(Q) Show that S_n is group under composition mapping.

(Solⁿ) S_n is set of all 1-1, onto mapping from set containing n elts to itself.

① closure $f \circ g \in S_n \quad \forall f, g \in S_n$ f is 1-1 & onto
 g is 1-1 & onto

② $f \circ (g \circ h) = (f \circ g) \circ h \quad \forall f, g, h \in S_n$

③ $\exists I \in S_n \quad I = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$

$$\text{s.t. } f \circ I = I \circ f = f \quad \forall f \in S_n$$

④ for each $f \in S_n$, as f is 1-1 onto mapping then f^{-1} is exists (it is also 1-1 & onto) (as f is 1-1 & onto)

$$\text{s.t. } f \circ f^{-1} = f^{-1} \circ f = I$$

then S_n is group w.r.t mapping composition.

Symmetric group S_2 of degree 2
 $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix} \right)$ $O(S_2) = 2! = 2 \times 1 = 2$

$$(12)^{-1} = (21) \\ I^{-1} = I$$

Symmetric group S_3 of degree 3
 S_3 denote set of all 1-1 onto functions from $\{1, 2, 3\}$ to itself. S_3 is group with six elts.
 The six elts are

$$I = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix} \right) \text{ or } \left[\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix} \right]$$

$$\alpha_1 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix} \right) = \alpha_2 (\alpha^2)$$

$$\beta_1 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix} \right), \beta_2 \stackrel{(\alpha\beta)}{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix} \right)}, \beta_3 \stackrel{(\alpha^2\beta)}{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix} \right)}$$

$S_3 = \{I, (123), (132), (23), (12), (13)\}$ is group with identity $I = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix} \right)$

$$O(S_3) = 3! = 3 \times 2 = 6$$

Inverse of each element of S_3

$$I^{-1} = I$$

$$(12)^{-1} = (21)$$

$$(13)^{-1} = (31)$$

$$(23)^{-1} = (32)$$

$$(123)^{-1} = (321) = (132)$$

$$(132)^{-1} = (231) = (123)$$

$$(12)(12)^{-1} = (12)(21)$$

$$= \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix} \right)$$

$$= \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix} \right)$$

$$= I_3$$

Abelian or Non-Abelian (S_3)

$$\alpha \beta \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix} \right) = (13)$$

$$\beta \alpha \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix} \right) = (12)$$

$$\alpha \beta \neq \beta \alpha$$

So S_3 is Non-Abelian grp

$$S_4 = 4! = 4 \times 3 \times 2 \times 1$$

$$= 24 \text{ elems}$$

$$O(S_5) = 5! = 5 \times 4 \times 3 \times 2 \times 1$$

$$= 120 \text{ elems}$$

$$S_4 = \{ I, (12), (13), (14), (23), (24), (34), (123), (124), (134), (234), (432), (431), (421), (321), (1234), (1243), (1423), (3241), (3421), (4321), (12)(34), (13)(24), (14)(23) \}$$

Cycle: A permutation $f \in S_n$ of length r is called

r -cycle

$$\text{ie } f = \begin{pmatrix} a_1 & a_2 & \dots & a_{r-1} & a_r \\ a_2 & a_3 & \dots & a_r & a_1 \end{pmatrix}$$

$$= \underbrace{(a_1, a_2, a_3, \dots, a_r)}_{r \text{ length}}$$

is called permutation of length r if $a_i \neq a_j, (i \neq j)$

example: $f = (1234) \in S_4$ is of length 4

$f = (1324)$ is of length 4

Transposition: A permutation $f \in S_n$ of length two (2 cycle) is called transposition.

eg $f = (13), f = (23), f = (34)$

(a) what is length of following permutations:

(132) is 3 cycle $132 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ in S_3

(23) is 2 cycle $(23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ in S_3

(1254) is 4 cycle in S_6

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 1 & 4 & 6 \end{pmatrix}$ length 4.

Product of two permutations

let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \in S_4$

then $(f \circ g)(1) = f(g(1)) = f(3) = 1$

$(f \circ g)(2) = f(g(2)) = f(2) = 4$

$(f \circ g)(3) = f(g(3)) = f(4) = 3$

$(f \circ g)(4) = f(g(4)) = f(1) = 2$

② $(134)(24) = (1342)$ in S_4

③ $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 9 & 8 & 5 & 6 & 2 \end{pmatrix} \in S_9$

can be expressed as

$f = (134)(2759)(68)$

Order of permutation

① $f = (a_1 a_2 \dots a_r) \in S_n$
then $o(f) = \text{length of } f = r$
 $\Rightarrow o(f) = r$

② $f = f_1 f_2 \dots f_k$ where $f_1, f_2, \dots, f_k$ are disjoint permutations
then $o(f) = \text{lcm}(o(f_1), o(f_2), \dots, o(f_k))$

(e.g.) $f = (1379)$
 $o(f) = 4$

③ $f = (135)(2467)$
 $o(f) = \text{lcm}(o(f_1), o(f_2))$
 $= \text{lcm}(3, 4)$
 $= 12$

④ $f = (1245)(36)$
 $o(f) = \text{lcm}(4, 2)$
 $= 4$