Case IV :- Equations that do not contain x :-

When the eq^n has the form

$$f(y,p)=0, \quad \text{——} (1)$$

and thus is solvable for p, it will give

$$\frac{dy}{dx}=\phi(y),$$

which is integrable.

If (1) is solvable for y, it will give

$$y=f(p)$$

which is Case III.

Example :- $y^2=a^2(1+p^2)$ —— (1)

Rewritting eq^n (1), $p=\pm\frac{\sqrt{y^2-a^2}}{a}$ or $\frac{dy}{dx}=\pm\frac{\sqrt{y^2-a^2}}{a}$.

$$\frac{a}{\sqrt{y^2-a^2}}\,dy=\pm\,dx$$

By Integrating, we get

$$a\log\{y+\sqrt{y^2-a^2}\}=\pm x-\log c$$

$$a\log\{y+\sqrt{y^2-a^2}\}+\log c=\pm x$$

$$\log[c\{y+\sqrt{y^2-a^2}\}^a]=\pm x$$

or

$$\{\log[c\{y+\sqrt{y^2-a^2}\}^a]\}^2=x^2$$

Exercise :- (1) $y=2p+3p^2$.

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Case V :- Equations that do not contain y :-

when the equation of the form

$$f(x, p) = 0 \quad \text{—} (1)$$

and it is solvable for p it will give

$$\frac{dy}{dx} = \phi(x),$$

which is integrable.

If eqn (1) is solvable for x, it will give

$$x = f(p)$$

which is Case (II).

Exercise :- (1) $x(1+p^2) = 1$
(2) $x^2 = a^2(1+p^2)$ } solve

Case VI :- Equations homogeneous in x and y :-

when eqn is homogeneous in x and y, it can be put in the form

$$F\left(\frac{dy}{dx}, \frac{y}{x}\right) = 0 \quad \text{—} (1)$$

It may be possible to solve eqn (1) for $\frac{dy}{dx}$, {Means $\frac{dy}{dx} = \frac{f(x,y)}{f(x,y)}$}

then take $y = vx$ and solve it. {Homogeneous ~~differential~~ eqn of 1st order & 1st degree}

or eqn (1) solvable for $\frac{y}{x}$, then we get

$$\frac{y}{x} = f(p) \Rightarrow y = x f(p)$$

which is same as Case (III) {solvable for y}.

Exercise :- (1) solve $y^2 + xyp - x^2p^2 = 0$
(2) $y = yp^2 + 2px$

Special type of 1st order but not first degree equations:-

Clairaut's Equation :-

The equation of the form

$$y = px + f(p) \quad \text{--- (1)}$$

is called Clairaut's Equation.

Solution of Clairaut's equation:-

Differentiation of eqn (1) with respect to 'x' gives

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or

$$\{x + f'(p)\}\frac{dp}{dx} = 0$$

Hence

$$x + f'(p) = 0 \quad \text{or} \quad \frac{dp}{dx} = 0$$

$$p = c \text{ (Constant)}$$

Now putting the value of p in equation (1).

we get

$$\boxed{y = cx + f(c)} \quad \text{--- (2)}$$

which is the solution of equation (1).

Example (1): Solve $y = (1+x)p + p^2$ --- (1)

rewrite equation (1) as

$$y = px + (1 + p^2) \quad \text{--- (2)}$$

Since equation (2) is in Clairaut's form.
hence solution of (1) is

$$\boxed{y = cx + 1 + c^2}$$

where c is an arbitrary Constant.

<u>Note</u>:- Some equations are not in Clairaut's from but Can be transform to Clairaut's form by using suitable transformation.

<u>Example</u>:- Solve $x^2(y - px) = yp^2$ ——(1)

Use transformation $u = x^2$ & $v = y^2$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dy} = 2y$$

$$\frac{dv}{du} = \frac{2y}{2x}\frac{dy}{dx} \Rightarrow \boxed{\frac{dy}{dx} = \frac{x}{y}\frac{dv}{du}}$$

Putting the value of $\frac{dy}{dx}$ in eqn (1)
we get

$$x^2\left(y - \frac{x^3}{y}\frac{dv}{du}\right) = y\left(\frac{dv}{du}\right)^2 \times \frac{x^2}{y^2}$$

$$\frac{1}{y}\left(y^2 - x^2\frac{dv}{du}\right) = \frac{1}{y}\left(\frac{dv}{du}\right)^2$$

$$y^2 - x^2\frac{dv}{du} = \left(\frac{dv}{du}\right)^2$$

or

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2 \quad [\text{as } x^2 = u \;\&\; y^2 = v]$$

Let $\frac{dv}{du} = P$

then

$$v = uP + P^2 \text{ ——(2)}$$

Since eqn (2) is in Clairaut's form
hence soln of (2) is

$$v = uc + c^2$$

soln of (1) is $\boxed{y^2 = cx^2 + c^2}$ Ans

Exercise :- (1) Solve $y = xp + \sin^{-1} p$

(2) Solve $e^{4x}(p-1) + e^{2y}p^2 = 0$ { $e^{4x} = u$ & $e^{2y} = v$ }

(3) $xy(y - px) = x + py$ { use $u = x^2$ & $v = y^2$ }

EXAMPLES ON CHAPTER III.

1. $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + 2y^2 - x^2 = 0.$

2. $y = p(x-b) + \frac{a}{p}.$

3. $xy^2(p^2+2) = 2py^3 + x^3.$

4. $y = -xp + x^4p^2.$

5. $p^2 - 9p + 18 = 0.$

6. $ayp^2 + (2x-b)p - y = 0.$

7. $y - px = \sqrt{1+p^2}\phi(x^2+y^2).$

8. $(xp-y)^2 = a(1+p^2)(x^2+y^2)^{\frac{3}{2}}.$

9. $(xp-y)^2 = p^2 - 2\frac{y}{x}p + 1.$

10. $3p^2y^2 - 2xyp + 4y^2 - x^2 = 0.$

11. $(x^2+y^2)(1+p)^2 - 2(x+y)(1+p)(x+yp) + (x+yp)^2 = 0.$

12. $(py+nx)^2=(y^2+nx^2)(1+p^2)$.

13. $y^2(1-p^2)=b$.

14. $(px-y)(py+x)=h^2p$.

15. $p^2+2py\cot x=y^2$.

16. $\left(p^2-\dfrac{1}{a^2-x^2}\right)\left(p-\sqrt{\dfrac{y}{x}}\right)=0$.

17. $x+\dfrac{p}{\sqrt{1+p^2}}=a$.

18. $y-2px=f(xp^2)$.

19. $xyp^2+p(3x^2-2y^2)-6xy=0$.

20. $p^3-4xyp+8y^2=0$.

21. $p^3-(x^2+xy+y^2)p^2+(x^3y+x^2y^2+xy^3)p-x^3y^3=0$.

22. $p^3+mp^2=a(y+mx)$.

23. $e^{3x}(p-1)+p^3e^{2y}=0$.

24. $\left(1-y^2+\dfrac{y^4}{x^2}\right)p^2-2\dfrac{y}{x}p+\dfrac{y^2}{x^2}=0$.

25. $y-(1+p^2)^{-\frac{1}{2}}=b$.

26. $y=px+\dfrac{m}{p}$.

27. $y=2px+y^2p^3$.