The Covering Relation.

Let P be a Poset. If $x, y \in P$, we say x is Covered by y, written as $x \prec y$ if $x < y$ & if $x \leq z < y$, then $x = z$.

Remark If $(P, \leq)$ is a finite Poset.
$x < y$ iff $\exists$ a finite Sequence of Covering relations $x \prec x_1 \prec x_2 \prec \cdots \prec x_n = y$.

example 1 $(\mathbb{N}, \leq) = P$ is a poset.
$x \prec y$ in $\mathbb{N}$ iff $y = x+1$

ex 2 In $(\mathbb{R}, \leq)$, $\nexists$
There are no Covering relations defined in $\mathbb{R}$

ex 3 If A is any set, $P = (\mathcal{P}(A), \subseteq)$ is a Poset. $X \prec Y$ in P iff
$Y = X \cup \{b\}$ where $b \in A - X$.

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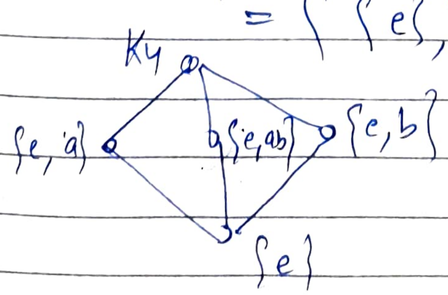
ex $\mathbf{2}$ = chain of 2 elts
$= \{0, 1\}$.

ex $\mathbf{3}$ = Antichain of 3 elts. o o o

ex K_4 = Klein's 4 group. $= \{e, a, b, ab\}$.
$a^2 = b^2 = e$, $a = a^{-1}$, $b = b^{-1}$.
$ab = ba$.

Sub K_4 = Set of all Subgroups of K_4
$= \{\{e\}, K_4, \{e,a\}, \{e,b\}, \{e,ab\}\}$.

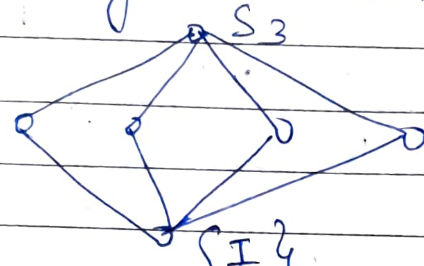


N-Sub K_4 = set of all normal subgroups of K_4 = Sub K_4.

ex S_3 = Symmetric group of order 3
$= \{I, (ab), (ac), (bc), (abc), (acb)\}$

Sub $S_3 = \{\{I\}, S_3, \{I, (ab)\}, \{I, (ac)\}, \{I, (bc)\}, \{I, (abc), (acb)\}\}$

Diagram of Sub S_3.



Sub S_3 is a lattice.

N-Sub S_3 = Set of all normal subgroups of ~~So~~ S_3
$= \{\{I\}, \{I, (abc), (acb)\}, S_3\}$

It is a chain.

Lemma: Let P & Q be finite ordered sets & let $\phi : P \to Q$ be a bijective function. Then TFAE.

(i) ϕ is an order isomorphism.

(ii) $x < y$ in P iff $\phi(x) < \phi(y)$ in Q

(iii) $x \prec y$ in P iff $\phi(x) \prec \phi(y)$ in Q.

Pf (i) $\Rightarrow$ (ii) Let $x < y$ in P
$\Rightarrow x \leq y$ & $x \neq y$

$x \leq y \Rightarrow \phi(x) \leq \phi(y)$ in Q (By (i))

Let $\phi(x) = \phi(y)$

Since ϕ is 1-1, $x = y$ Contradiction

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$\therefore$ $\phi(x) \neq \phi(y)$

$\Rightarrow$ $\phi(x) < \phi(y)$

Now let $\phi(x) < \phi(y)$ in Q

$\Rightarrow$ $\phi(x) \leq \phi(y)$ & $\phi(x) \neq \phi(y)$

$\Rightarrow$ $x \leq y$ (By (i))

Suppose $x = y$

Since ϕ is well defined, $\therefore$ $\phi(x) = \phi(y)$

This is a contradiction

$\therefore$ $x \neq y$ $\Rightarrow$ $x < y$

$\therefore$ $x < y$ in P iff $\phi(x) < \phi(y)$ in Q

$\therefore$ (i) $\Rightarrow$ (ii)

Let (ii) be given.

To prove (i) let $x \leq y$ in P

$\Rightarrow$ Either $x < y$ or $x = y$

$\Rightarrow$ $\phi(x) < \phi(y)$ (By (ii)) or $\phi(x) = \phi(y)$

$\Rightarrow$ $\phi(x) \leq \phi(y)$ in Q

Now let $\phi(x) \leq \phi(y)$ in Q

$\Rightarrow$ $\phi(x) < \phi(y)$ or $\phi(x) = \phi(y)$

$\Rightarrow$ $x < y$ (By (ii)) or $x = y$ ($\because$ ϕ is H)

$\Rightarrow$ $x \leq y$ in P

$\therefore$ (i) is proved.

To prove (ii) $\Leftrightarrow$ (iii)

Let $x < y$ in P iff $\phi(x) < \phi(y)$ in Q.

Let $x \prec y$ in P

$\Rightarrow$ $x < y$ in P

By (ii), $\phi(x) < \phi(y)$ in Q

Suppose $\exists\, t \in Q$ s.t. $\phi(x) \leq t < \phi(y)$

Since $\phi : P \to Q$ is bijective,

$\exists$ a unique elt. $z \in P$ s.t. $\phi(z) = t$

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$\therefore \phi(x) \leq \phi(z) < \phi(y)$

$\phi(z) < \phi(y) \Rightarrow z < y$

& $\phi(x) \leq \phi(z) \Rightarrow \phi(x) < \phi(z)$ or $\phi(x) = \phi(z)$

$\therefore$ By (ii), $x < z$ or $x = z$ ($\because \phi$ is 1-1)

$\Rightarrow x \leq z < y$

But $x \prec y$

$\Rightarrow x = z$

$\Rightarrow \phi(x) = \phi(z) = t.$

$\therefore \phi(x) \prec \phi(y)$

Now let $\phi(x) \prec \phi(y)$

$\Rightarrow \phi(x) < \phi(y)$

$\Rightarrow x < y$ (By (ii))

Suppose $\exists z \in P$ s.t. $x \leq z < y$

$\Rightarrow \phi(x) \leq \phi(z) < \phi(y)$ (By (ii))

But $\phi(x) \prec \phi(y)$

$\Rightarrow \phi(x) = \phi(z)$

$\Rightarrow x = z$ (~~By~~ $\because \phi$ is 1-1)

$\therefore x \prec y$

$\therefore$ (iii) is true

Conversely let (iii) be given.

& let $\boxed{x < y \text{ in } P}$

$\Rightarrow \exists$ a sequence of covering relations.

$x = x_0 \prec x_1 \prec x_2 \prec \dots \prec x_n = y$

By (iii) $\phi(x) = \phi(x_0) \prec \phi(x_1) \prec \dots \prec \phi(x_n) = \phi(y)$

$\Longrightarrow \boxed{\phi(x) < \phi(y) \text{ in } Q}$.

Now let $\phi(x) < \phi(y)$ in Q

$\Rightarrow \exists$ a sequence of covering relations

$\phi(x) = t_0 \prec t_1 \prec t_2 \prec \dots \prec t_n = \phi(y)$

Since ϕ is bijective $\therefore \exists$ unique set of elts

$z_1, \dots, z_n \in P$ s.t. $\phi(z_i) = t_i$

By (iii) $x_0 \prec z_1 \prec z_2 \prec \dots \prec z_n = y$

$\Rightarrow x < y$ in P. Hence (ii) $\Leftrightarrow$ (iii)