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09:16

For two finite subgroups H and K of a group, $|HK| = \frac{|H| |K|}{|H \cap K|}$, where

$$HK = \{hk \mid h \in H, k \in K\}$$

Proof:

$$\text{Now } HK = \{hk \mid h \in H, k \in K\}$$

$$|H| \quad |K|$$

$$a \in H, a \in K$$

$$hk \rightarrow |H| |K| \quad H \cap K = \{e\}, |H \cap K| = 1$$

the set HK has $|H| |K|$ products, but not all of these products need represent distinct group elements. That is, we may have

$$hk = h'k', \text{ where } h \neq h' \text{ and } k \neq k'$$

for every $t \in H \cap K$, the product hk can be written as $hk = (ht)(t^{-1}k)$.

So every group element in HK is represented by at least $|H \cap K|$ products in HK .

$$\text{Now, if } hk = h'k'$$

$$\Rightarrow h^{-1}h' = k(k')^{-1} = t \in H \cap K$$

$$\Rightarrow h' = ht \text{ and } k' = t^{-1}k, \\ \text{where } t \in H \cap K.$$

Thus, each element in HK is represented by exactly $|H \cap K|$ products,

$$\therefore |HK| = \frac{|H||K|}{|H \cap K|}$$

$$\boxed{hk = (ht)(t^{-1}k) \rightarrow t \rightarrow |H \cap K|}$$

"Hence proved".

Q. A group of order 75 can have atmost one subgroup of order 25.

Soln let G be a group and $|G| = 75$.

let there are two subgroups H and K of order 25 of group G .

$\therefore H \cap K$ is a subgroup of H .

$\Rightarrow |H \cap K|$ divides $|H|$

$\Rightarrow |H \cap K| = 1 \text{ or } 5 \text{ or } 25$ ——— ①

Case - I
If $|H \cap K| = 1$,

$$\text{then } |HK| = \frac{|H||K|}{|H \cap K|} = \frac{(25)(25)}{1} = 625$$

$$\therefore |HK| > |G|$$



Note: HK is a subgroup of G if H and K are subgroups of G .

Case-II: If $|H \cap K| = 5$.

$$|HK| = \frac{|H||K|}{|H \cap K|} = \frac{25 \cdot 25}{5} = 125$$

$$\therefore |HK| > |G| \rightarrow \leftarrow$$

Case-III: If $|H \cap K| = 25$.

$$\therefore |H| = |K| = 25, \quad \& \quad |H \cap K| = 25.$$

$$\therefore H = K.$$

\rightarrow there is only one subgroup of order 25.

Q. 16 Prove that a finite group is the union of proper subgroups if and only if the group is not cyclic.

Solⁿ

Let G be a finite group, $|G| = n$.

Suppose that G is the union of proper subgroups H_1, H_2, \dots, H_k .

$$\therefore G = H_1 \cup H_2 \cup \dots \cup H_k \quad \text{--- ①}$$

TS G is not cyclic.

Assume that G is cyclic.

$$\Rightarrow G = \langle a \rangle \quad \text{--- ②}$$

$$\text{Now } a \in G \Rightarrow a \in H_i \text{ for some } 1 \leq i \leq k$$

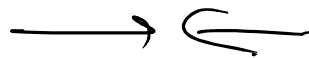
$$\Rightarrow a^2 \in H_i, a^3 \in H_i, \dots$$

$$\Rightarrow \langle a \rangle \subseteq H_i$$

$$\Rightarrow G \subseteq H_i$$

$$\Rightarrow G = H_i \quad (\because H_i \subseteq G) \text{ for some } 1 \leq i \leq k.$$

$\Rightarrow H_i$ is not the proper subgroup of G



Conversely (C) If G is not cyclic,

$$\text{then } a_1 \in G \Rightarrow H_1 = \langle a_1 \rangle$$

$$a_2 \in G \text{ such that } a_2 \neq a^k \text{ for } k \in \mathbb{Z}.$$

$$\text{then we can have subgroup } H_2 = \langle a_2 \rangle$$

Continuing in this manner, we will have

$$H_1 = \langle a_1 \rangle, H_2 = \langle a_2 \rangle, \dots, H_k = \langle a_k \rangle$$

$\therefore G$ is finite, so there are finite no. of subgroups of G .

So we can write, $G \cong H_1 \cup H_2 \cup \dots \cup H_k$.
where H_1, H_2, \dots, H_k are proper subgroups.