09 October 2020 09:16

Thm For two finite subgroups H and K of a group, $|HK| = \frac{|H||K|}{|H \cap K|}$, where $HK = \{hk \mid h \in H, k \in K\}$

Proof: Now $HK = \{hk \mid h \in H, k \in K\}$.

$|H|$ $|K|$

$a \in H$, $a \in K$.

$hk \to |H||K|$. $H \cap K = \{e\}$, $|H \cap K| = 1$

the set HK has $|H||K|$ products, but not all of these products need represent distinct group elements. That is, we may have

$hk = h'k'$, where $h \neq h'$ & $k \neq k'$

for every t in $H \cap K$, the product hk can be written as $hk = (ht)(t^{-1}k)$.

So every group element in HK is represented by at least $|H \cap K|$ products in HK.

Now, if $hk = h'k'$

$\Rightarrow h^{-1}h' = k(k')^{-1} = t \in H \cap K$.

$\Rightarrow \quad h' = ht$ and $k' = t^{-1}k$,
where $t \in H \cap K$.

Thus, each element in HK is represented by exactly $|H \cap K|$ products.

$\boxed{hk = (ht)(t^{-1}k) \quad t \to |H \cap K|}$

$$\therefore \quad |HK| = \frac{|H||K|}{|H \cap K|}$$

"Hence Proved".

Q. A group of order 75 can have atmost one subgroup of order 25.

Soln

Let G be a group and $|G| = 75$.

Let there are two subgroups H and K of order 25 of group G.

$\because$ $H \cap K$ is a subgroup of H.

$\Rightarrow$ $|H \cap K|$ divides $|H|$

$\Rightarrow$ $|H \cap K| = 1$ or 5 or 25 —— (1)

Case-I
If $|H \cap K| = 1$,

then $|HK| = \dfrac{|H||K|}{|H \cap K|} = \dfrac{(25)(25)}{1} = 625$

$\therefore |HK| > |G|$

$\longrightarrow \Leftarrow$

$\boxed{\text{Note:} \rightarrow HK \text{ is a subgroup of } G \text{ if } H \text{ and } K \text{ are subgroups of } G}$

Case-II: If $|H \cap K| = 5$.

$$|HK| = \frac{|H||K|}{|H \cap K|} = \frac{25 \cdot 25}{5} = 125$$

$\therefore |HK| > |G| \longrightarrow \Leftarrow$

Case-III: If $|H \cap K| = 25$

$\because |H| = |K| = 25$, & $|H \cap K| = 25$.

$\therefore H = K$

$\rightarrow$ there is only one subgroup of order 25.

Q. 16 Prove that a finite group is the union of proper subgroups if and only if the group is not cyclic.

Sol: Let G be a finite group, $|G| = n$.

Suppose that G is the union of proper subgroups $H_1, H_2, \ldots, H_k$.

$\therefore G = H_1 \cup H_2 \cup \cdots \cup H_k$ ——— (1)

TS G is not cyclic

Assume that G is cyclic

$\Rightarrow G = \langle a \rangle$ ——— (2)

Now $a \in G \Rightarrow a \in H_i$ for some $1 \le i \le k$

$\Rightarrow a^2 \in H_i,\ a^3 \in H_i, \ldots$

$\Rightarrow \langle a \rangle \subseteq H_i$

$\Rightarrow G \subseteq H_i$

$\Rightarrow G = H_i$ $(\because H_i \subseteq G)$, for some $1 \le i \le k$.

$\Rightarrow H_i$ is not the proper subgroup of G

$\longrightarrow \Leftarrow$

Conversely ($\Leftarrow$) If G is not cyclic,

then $a_1 \in G \Rightarrow H_1 = \langle a_1 \rangle$

$a_2 \in G$ such that $a_2 \neq a^k\ \forall\ k \in \mathbb{Z}$.

then we can have subgroup $H_2 = \langle a_2 \rangle$

Continuing in this manner, we will ...

$H_1 = \langle a_1 \rangle$, $H_2 = \langle a_2 \rangle$, --- $H_k = \langle a_k \rangle$.

$\because$ G is finite, so there are finite no. of subgroups of G.

So we can write. $G = H_1 \cup H_2 \cup --- \cup H_k$.
where $H_1, H_2, ---, H_k$ are proper subgroups.