6.2 The matrix of a linear transformation

Theorem 6.4 let $B = \{v_1, v_2, \cdots v_n\}$ be an ordered basis for a vector space V. let W be a vectorspace and let $w_1, w_2, \cdots w_n$ be any (n) not necessarily distinct vectors in W. Then there is one and only one L.T $T: V \to W$ satisfying $T(v_1) = w_1, T(v_2) = w_2 \cdots T(v_n) = w_n$.

proof: let v be any vector in V.
Since $B: \{v_1, v_2, \dots v_n\}$ be an ordered basis for V then there exits unique scalars $a_1, a_2, \cdots, a_n \in \mathbb{R}$
s.t $v = a_1 v_1 + \cdots + a_n v_n$
Define a function $T: V \to W$ by
$T(v) = a_1 w_1 + \cdots + a_n w_n$ — (1)

To show: T is L.T
let x & y be vectors in V then
$x = b_1 v_1 + \cdots + b_n v_n$
and $y = c_1 v_1 + \cdots + c_n v_n$
for some unique b_i's & c_i's in $\mathbb{R}$
Then by (1), we have
$T(x) = b_1 w_1 + \cdots + b_n w_n$
$T(y) = c_1 w_1 + \cdots + c_n w_n$
and $T(x) + T(y) = (b_1 w_1 + \cdots + b_n w_n) + (c_1 w_1 + \cdots + c_n w_n)$
$= (b_1 + c_1) w_1 + \cdots + (b_n + c_n) w_n$ — (2)

and $x + y = (b_1 v_1 + \cdots + b_n v_n) + (c_1 v_1 + \cdots + c_n v_n)$
$= (b_1 + c_1) v_1 + \cdots + (b_n + c_n) v_n$
$\therefore T(x+y) = (b_1 + c_1) w_1 + \cdots + (b_n + c_n) w_n$ — (3)
from (3) & (4), we get
$T(x+y) = T(x) + T(y)$

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Now, for any scalar, $c \in \mathbb{R}$

$$c x = c(b_1 v_1 + b_2 v_2 + \cdots + b_n v_n)$$
$$= (c b_1) v_1 + \cdots + (c b_n) v_n$$

$$\Rightarrow T(cx) = (c b_1) w_1 + \cdots + (c b_n) w_n \quad [\text{by } ①]$$
$$= c(b_1 w_1) + \cdots + c(b_n w_n)$$
$$= c(b_1 w_1 + \cdots + b_n w_n)$$
$$= c\,T(x)$$

$\therefore$ T is L. Transformation

Uniqueness Let $L: V \to W$ be another L.T

s.t $L(v_1) = w_1,\ L(v_2) = w_2 \cdots L(v_n) = w_n$

If $v \in V$

then $v = a_1 v_1 + \cdots + a_n v_n$

for unique scalars $a_1, a_2, \ldots, a_n \in \mathbb{R}$

But $L(v) = L(a_1 v_1 + \cdots + a_n v_n)$

$$= a_1 L(v_1) + \cdots + a_n L(v_n) \quad (\because L \text{ is L.T})$$
$$= a_1 w_1 + \cdots + a_n w_n$$
$$= T(v)$$

$\Rightarrow L = T$

$\therefore$ T is uniquely L.T

Example (14) Suppose $L: \mathbb{R}^3 \to \mathbb{R}^2$ is L.T

with $L([1,-1,0]) = [2,1]$

$L([0,1,-1]) = [-1,3]$

$L([0,1,0]) = [0,1]$

find $L([1,1,2])$

Also formula for $L([x,y,z])$ for any $[x,y,z] \in \mathbb{R}^3$

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Solution: Let $v = [-1, 1, 2]$

$v_1 = [1, -1, 0]$

$v_2 = [0, 1, -1]$

$v_3 = [0, 1, 0]$

to show: there are constants a_1, a_2 & a_3 s.t

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 \quad — (1)$$

which gives augmented matrix as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 2 \end{array}\right] \quad — (A)$$

whose reduced row echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

this gives $a_1 = -1$, $a_2 = -2$, $a_3 = 2$

So, we get

$$v = -v_1 - 2v_2 + 2v_3 \quad \text{[write (1) equation]}$$

$$\Rightarrow L(v) = L(-v_1 - 2v_2 + 2v_3)$$

$$= L(-v_1) - 2L(v_2) + 2L(v_3)$$

$$= (-1)L(v_1) - 2L(v_2) + 2L(v_3)$$

$$= -[2, 1] - 2[-1, 3] + 2[0, 1] \quad \text{(given in question)}$$

$\therefore L([-1, 1, 2]) = [0, -5]$

To find $L([x, y, z])$ for any $[x, y, z] \in \mathbb{R}^3$

Do same as (A)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ -1 & 1 & 1 & y \\ 0 & -1 & 0 & z \end{array}\right]$$

whose reduced row echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & -z \\ 0 & 0 & 1 & +x+y+z \end{array}\right]$$

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$\Rightarrow \cancel{L([x+y,z])}$

$= \cancel{L\,x}$

Thus, $[x,y,z] = x v_1 - z v_2 + (x+y+z) v_3$

$\Rightarrow L([x,y,z]) = L(x v_1 - z v_2 + (x+y+z) v_3)$

$= x L(v_1) - z L(v_2) + (x+y+z) L(v_3)$

$= x[2,1] - z[-1,3] + (x+y+z)\cdot[0,1]$

[given in question]

$= [2x+z, 2x+y-2z]$ (Hence done)

The Matrix of Linear Transformation

Let V and W be non-trivial vector spaces with $\dim V = n$ & $\dim W = m$

Let $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_m\}$ be ordered bases for V and W.

let $T: V \to W$ be L.T

For each $v \in V$, the co-ordinate vector for v & $T(v)$ are $[v]_B$ and $[T(v)]_C$ respect the ordered bases B and C, respectively.

Find $m \times n$ matrix $A = (a_{ij})$ where $1 \le i \le m$, $1 \le j \le n$

s.t $A[v]_B = [T(v)]_C$ holds for all vector $v \in V$ — (1)

equation (1) holds for all vectors in V, it must hold, in particular, for basis vectors in B

i.e $A[v_1]_B = [T(v_1)]_C$

$A[v_2]_B = [T(v_2)]_C$ — (2)

$\vdots \qquad \vdots$

$A[v_n]_B = [T(v_n)]_C$

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Since B & C are ordered basis we, have

$$[v_1]_B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad [v_2]_B = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots\cdots [v_n]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$\therefore$
$$A[v_1]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$\vdots$$

$$A[v_n]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Substituting these result in (2)

we get

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = [T(v_1)]_C$$

$$\vdots$$

$$\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = [T(v_n)]_C$$

$\therefore$ Successive columns of A are co-ordinates vectors of $T(v_1), \ldots T(v_n)$ with respect to ordered basis C.

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Thus, the matrix A given by
$A = [[T(v_1)]_C \ [T(v_2)]_C \ \cdots \ [T(v_n)]_C]$
matrix of T relative to bases B and C
will denote by the A_{BC} or $[T]_{BC}$
Thus, $A_{BC} = [[T(v_1)]_C \ [T(v_2)]_C \ \cdots \ [T(v_n)]_C]$

From ①, A_{BC} satisfies the property
$A_{BC}[v]_B = [T(v)]_C$ for all $v \in V$
∴ from above result, we get the following theorem

Theorem (6.5) Let V and W be non-trivial vector spaces with $\dim(V) = n$ & $\dim(W) = m$
let $B = \{v_1, v_2, \ldots v_n\}$
$C = \{w_1, w_2, \ldots w_m\}$
be ordered bases for V and W respectively.
let $T: V \to W$ be L.T
then there is unique $m \times n$ matrix A_{BC} s.t
$A_{BC}[v]_B = [T(v)]_C$ for all $v \in V$
Also, for $1 \le i \le n$, the ith column of $A_{BC} = [T(v_i)]_C$

(proof) (Above proved)

Example ⑯ Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be L. operator (linear operator)
given by $T([x_1, x_2, x_3]) = [3x_1 + x_2, x_1 + x_3, x_1 - x_3]$
Find matrix for T with respect to standard basis for $\mathbb{R}^3$

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Solution: The standard basis for $\mathbb{R}^3$ is $B = \{e_1 = [1,0,0],\ e_2 = [0,1,0],\ e_3 = [0,0,1]\}$

given $T([x_1, x_2, x_3]) = [3x_1 + x_2,\ x_1 + x_3,\ x_1 - x_3]$ — (1)

we get, using (1)

$T(e_1) = [3, 1, 1]$

$T(e_2) = [1, 0, 0]$

$T(e_3) = [0, 1, -1]$

Since the coordinate vector of any element $[x_1, x_2, x_3]$ in $\mathbb{R}^3$ with respect to standard basis $\{e_1, e_2, e_3\}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

we have

$$[T(e_1)]_B = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$[T(e_2)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(e_3)]_B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Thus, the matrix A_{BB} for T with respect to standard basis is

$$A_{BB} = \left[[T(e_1)]_B \ [T(e_2)]_B \ [T(e_3)]_B \right]$$

$$= \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

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Finding the New Matrix for L.T After change of Basis

Theorem (6.6) Let V and W be two non-trivial finite dimensional vector spaces with ordered bases B and C, respectively.
Let $T: V \to W$ be L.T with matrix A_{BC} with respect to bases B and C.
Suppose D and E are other ordered bases for V and W respectively.
Let P be transition matrix from B to D.
Let Q be transition matrix from C to E.
then matrix A_{DE} for T with respect to bases D and E given by $A_{DE} = Q A_{BC} P^{-1}$.

i.e (we have)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_B} \arrow[r, "A_{BC}", "\substack{\text{matrix}\\ \text{for } T \text{ using}\\ B,\, C}"'] \arrow[d, "\substack{\text{Transition}\\ \text{matrix } P}"'] & {[T(v)]_C} \arrow[d, "\substack{\text{Transition}\\ \text{matrix } Q}"] \\
{[v]_D} \arrow[r, "\substack{\text{matrix for } T\\ \text{using } D,\, E}"'] & {[T(v)]_E}
\end{tikzcd}$$

Example (20) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be Linear operator given by $T([a, b, c]) = [-2a+b, -b-c, a+3c]$

(a) Find matrix A_{BB} for T with respect to standard basis $B = \{e_1, e_2, e_3\}$ for $\mathbb{R}^3$

(b) Use part (a) to find the matrix A_{DE} with respect to bases
$D = \{[15, -6, 4], [2, 0, 1], [3, -1, 1]\}$

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and $E = \{[1,-3,1], [0,3,-1], [2,-2,1]\}$

Solution

(a) we have $T(e_1) = [2,0,1]$, $T(e_2) = [1,-1,0]$, $T(e_3) = [0,-1,3]$ } we get by using $T([a,b,c]) = [-2a+b, -b-c, a+3c]$

Using each of these vectors as columns matrix $A_{BB} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix}$

(b) Find A_{DE}

Using relation $A_{DE} = Q A_{BB} P^{-1}$ — (1)

where P is transition matrix from B to D.

and Q is transition matrix from B to E.

Since P^{-1} is transition matrix from D to B and B is standard basis for $\mathbb{R}^3$, is given by

$$P^{-1} = \begin{bmatrix} 15 & 2 & 3 \\ -6 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix}$$

To find Q

Q^{-1}, the transition matrix from E to B given by $\begin{bmatrix} 1 & 0 & 2 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

Now $Q = (Q^{-1})^{-1} = \begin{bmatrix} 1 & -2 & -6 \\ 1 & -1 & -4 \\ 0 & 1 & 3 \end{bmatrix}$

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Hence, using equation (1), we have

$$A_{DE} = Q\, A_{BB}\, P^{-1}$$

$$= \begin{bmatrix} 1 & 2 & -6 \\ 1 & -1 & -4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 15 & 2 & 3 \\ -6 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -202 & -32 & -43 \\ -146 & -23 & -31 \\ 83 & 14 & 18 \end{bmatrix}$$

6.3 Linear operators & Similarity

Theorem 6.7 Let V be finite-dimensional V.S with ordered bases B and C. Let T be L.O on V.
Then matrix A_{BB} for T with respect to the basis B is similar to the matrix A_{CC} for T with respect to the basis C.
More specifically, if P is transition matrix from B to C then $A_{BB} = P^{-1} A_{CC} P$.

(proof) By theorem 6.6 (above)
we get $A_{CC} = P A_{BB} P^{-1}$ where P is transition matrix from B to C.
$\Rightarrow A_{BB} = P^{-1} A_{CC} P$

$\therefore$ A_{BB} & A_{CC} are similar.

i.e (we have)

$$\begin{tikzcd}
{[v]_B} \arrow[r, "A_{BB}", "\text{matrix for } T \text{ using } B"'] & {[T(v)]_B} \arrow[d, "\text{Transition matrix } P"] \\
{[v]_C} \arrow[u, "\text{Transition matrix } P^{-1}"] \arrow[r, "A_{CC}", "\text{matrix for } T \text{ using } C"'] & {[T(v)]_C}
\end{tikzcd}$$

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Example (22) Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be linear operator given by $L([x, y]) = [2x-y, x-3y]$ find matrix for L with respect to basis $\{[4,-1], [-7,2]\}$ using method of similarity.

(Solution) Standard basis $B = \{e_1, e_2\}$ for $\mathbb{R}^2$

given $L([x,y]) = [2x-y, x-3y]$ — (1)

we have

$L(e_1) = L([1,0]) = [2,1]$ (using (1))

$L(e_2) = L([0,1]) = [-1,-3]$ (using (1))

Thus, matrix for L with respect to basis B is

$$A_{BB} = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$$

Next, we find transition matrix P from the standard basis B to basis $C = \{[4,-1], [-7,2]\}$

The transition matrix from C to B, given by

$$P^{-1} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$$

Hence, the matrix A_{CC} for L with respect to basis C is given by

$$A_{CC} = P A_{BB} P^{-1}$$

$$= \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 67 & -123 \\ 37 & -68 \end{bmatrix}$$

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(11)

Theorem 6.8 Let V_1, V_2 & V_3 be non-trivial finite dimensional vector space with ordered bases B, C & D respectively.
Let $T_1 : V_1 \to V_2$ be L.T
$T_2 : V_2 \to V_3$ be L.T
with matrix A_{BC} with respect to bases B & C
and matrix A_{CD} with respect to bases C & D, respectively.
Then the matrix A_{BD} for the composition L.T $T_2 \circ T_1 : V_1 \to V_3$ with respect to bases B & D given by $A_{BD} = A_{CD} A_{BC}$.

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