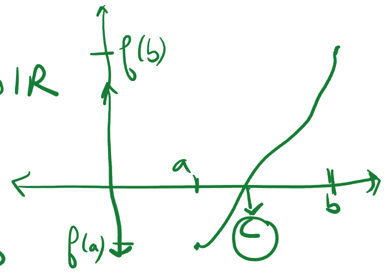
9th October 2020

Location of roots theorem

$f : [a,b] \xrightarrow{\text{cts}} \mathbb{R}$



such that $f(a) < 0 < f(b)$

then $\exists\, c \in (a,b) : f(c) = 0$

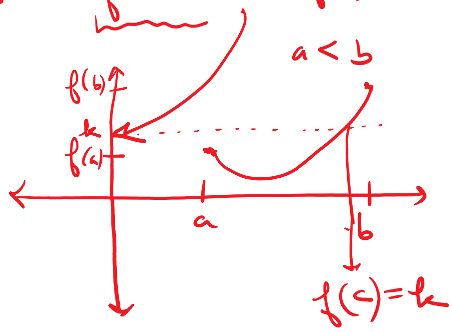
Theorem 5.3.7 **Bolzano's Intermediate value theorem**

Let I be any interval & $f : I \to \mathbb{R}$ be cts on I

If $a, b \in I$ & if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$

then there a point $c \in I$ between a & b such that $f(c) = k$

$a < b$



Proof Let $a < b$ (the case $a > b$ similar)

Define $g : [a,b] \to \mathbb{R}$ as

$$g(x) = f(x) - k \quad \forall x \in [a,b]$$

Then, $g(a) = f(a) - k < 0$ ($\because f(a) < k$) & $g(b) = f(b) - k > 0$ ($\because k < f(b)$)

$\therefore g(a) < 0 < g(b)$, $g : [a,b] \to \mathbb{R}$ & g is cts on $[a,b]$

$\therefore$ by location of roots theorem, $\exists\, c \in (a,b) :$ ($\because f$ is cts & $x \to k$ is cts)

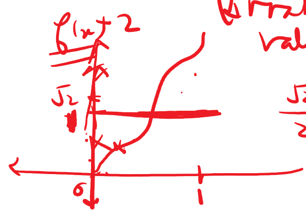
$$g(c) = 0$$

Wed 14/10 5.3 only 1 question

$$\Rightarrow f(c) - k = 0$$

$$\Rightarrow f(c) = k \text{ for some } c \text{ lying between } a \text{ \& } b$$

Q 17 If $f : [0,1] \to \mathbb{R}$ is cts and has only rational values (respectively irrational values) must f be constant? Prove your assertion.

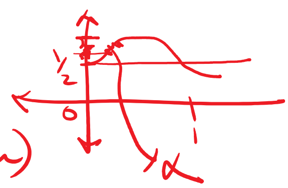


$\frac{\sqrt{2}}{2} \approx .7$

Sol Suppose f is not constant

$\Rightarrow \exists\, x, y \in I = [0,1] :$

$$f(x) \neq f(y)$$



Let $f(x) < f(y)$ (case $f(x) > f(y)$ is similar)

$\exists$ an irrational no. k : $f(x) < k < f(y)$ (by density thm)

& f is cts on I ; $x, y \in I$

& f is cts on I ; $x, y \in I$

By Intermediate Value theorem, $\exists$ c lying between x & y such that $f(c) = k$

rational ← → irrational

This is a contradiction as $f(c)$ is rational (given.....) & k is irrational

$\therefore$ f is a constant function

Cor. 5.3.8 Let $I = [a,b]$ be a closed, bdd interval & $f : I \to \mathbb{R}$ be cts on I. If $k \in \mathbb{R}$ is any no satisfying

$$\inf f(I) \le k \le \sup f(I) \quad \text{——— (1)}$$

then $\exists$ a no $c \in I$: $f(c) = k$

Proof

$f : I \xrightarrow{\text{cts}} \mathbb{R}$. I is closed & bdd

$\therefore$ by max-min thm, $\exists$ $x, y \in I$:

$$f(x) = \inf f(I) \quad \& \quad f(y) = \sup f(I)$$

$\therefore$ from (1) we get, $f(x) \le k \le f(y)$ where $x, y \in I$

If $k = x$ or $k = y$ we get $f(x) = k$ or $f(y) = k$ & result is proved

Otherwise, $f(x) < k < f(y)$, $f : I \xrightarrow{\text{cts}} \mathbb{R}$, $x, y \in I$

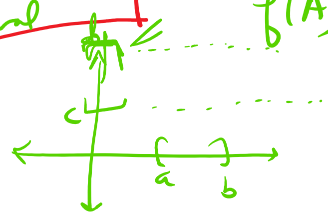
By I.V.T., $\exists$ $c \in I$: $f(c) = k$

$\boxed{\text{Continuous Image of a closed \& bdd interval is also a closed \& bdd interval}}$

$f : A \to \mathbb{R}$

$f(A)$

$f : [a,b] \to f[a,b]$ $\qquad f([a,b]) = [c,d]$



Thm 5.3.9 Let I be closed & bdd interval & $f : I \to \mathbb{R}$ be cts on I. Then the set

$$f(I) = \{ f(x) \mid x \in I \}$$

is a closed & bdd interval

$m = \inf f(I)$

$M = \sup f(I)$

Proof Let $f : I \xrightarrow{\text{cts}} \mathbb{R}$, $I = [a,b]$

By bddness theorem, $f(I)$ is a bdd set

$\Rightarrow$ Inf $f(I)$ & sup $f(I)$ exist

Let $m = \text{Inf } f(I)$ & $M = \sup f(I)$

$f(I)$ **Claim:** $f(I) = [m, M]$

$A \subseteq B$
$A \supseteq B$

for any $f(x) \in f(I)$ $\Rightarrow$ $m \leq f(x) \leq M$

$\Rightarrow f(x) \in [m, M]$

$\therefore f(I) \subseteq [m, M]$ ——— (1)

Let $k \in [m, M] \Rightarrow m \leq k \leq M$

i.e. $\text{Inf } f(I) \leq k \leq \sup f(I)$

(by cor 5.3.8) $\exists\, c \in I : f(c) = k$

$\Rightarrow k \in f(I)$

$\therefore [m, M] \subseteq f(I)$ ——— (2)

$\therefore$ (1) & (2) $\Rightarrow f(I) = [m, M]$

2.5.1 Characterization Theorem *If S is a subset of $\mathbb{R}$ that contains at least two points and has the property*

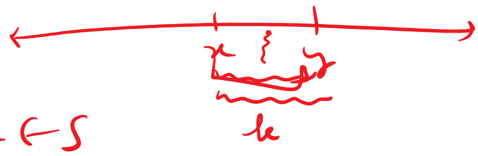
(1) *if* $x, y \in S$ *and* $x < y$, *then* $[x, y] \subseteq S$,

then S is an interval.

Proof There are four cases to consider: (i) S is bounded, (ii) S is bounded above but

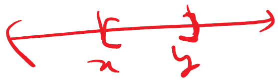
$\{1, 2, 3\}$

$x < y$ $x, y \in S$



$k \in [x, y] \Rightarrow k \in S$

$[x, y] \subseteq S$



$\subseteq S$

Theorem 5.3.10 **Preservation of Intervals**

I be any interval & $f : I \to \mathbb{R}$ be cts on I then $f(I)$ is an Interval

$f : [a, b] \xrightarrow{cts} \mathbb{R}$
$\Rightarrow f(I) = [m, M]$

Proof Given: $f : I \xrightarrow{cts} \mathbb{R}$

To show: $f(I)$ is an Interval (We'll use

To show: $f(I)$ is an Interval (We'll use characterization prop.)

Let $\alpha, \beta \in f(I)$, $\alpha < \beta$ (we'll show $[\alpha, \beta] \subseteq f(I)$)

As $\alpha, \beta \in f(I)$ $\therefore \exists\, a, b \in I : f(a) = \alpha$ & $f(b) = \beta$

Let $k \in [\alpha, \beta]$

$\Rightarrow \alpha \le k \le \beta$

$\Rightarrow f(a) \le k \le f(b)$ for some $a, b \in I$

$\therefore$ by I.V.T., $\exists\, c \in I \therefore f(c) = k$

$\Rightarrow k \in f(I)$ $\therefore [\alpha, \beta] \subseteq f(I)$

$\therefore$ for any $\alpha, \beta \in f(I) = S$, $\alpha < \beta$; $[\alpha, \beta] \subseteq f(I) = S$

$\Rightarrow \underbrace{f(I) \text{ is an Interval}}$ (by charact. thm)