(12.1)

(Ch-12) $x^2+y^2=z^2$

Task of find ~~eaidentifying~~ all solutions in the positive integers of equation $x^2+y^2=z^2$

(Defn 12.1) A Pythagorean triple is the set of three integers x, y, z such that $x^2+y^2=z^2$, the triple is said to be primitive if $\gcd(x, y, z)=1$

(e.g) 3, 4, 5 are Pythagorean triples, 5, 12, 13
(it has relationship to Pythagoras form) (some how)

(lemma 1) (Not proof) If x, y, z is a primitive Pythagorean triple, then one of the integers x or y is even, while the other is odd.

(lemma 2) (Not proof) If $ab=c^n$, where $\gcd(a, b)=1$ then a & b are nth powers;
i.e $\exists$ positive integers a_1, b_1 for which
$a=a_1^n$, $b=b_1^n$.

(Theorem 12.1) All solutions of Pythagorean Equations
$$x^2+y^2=z^2$$
satisfying the conditions $\gcd(x, y, z)=1$, $2|x$,
$x>0, y>0, z>0$
are given by formula.
$$x=2st, \quad y=s^2-t^2, \quad z=s^2+t^2$$
for integers $s>t>0$ such that $\gcd(s, t)=1$
and $s \not\equiv t \pmod 2$

(Pf) let x, y, z are positive primitive Pythagorean triple.
by lemma (1) take x even, y & z both odd
we get $z-y$, $z+y$ are even integers.
say $z-y=2u$ & $z+y=2v$.

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Now equation $x^2+y^2=z^2$

$x^2 = z^2-y^2 = (z-y)(z+y)$

and $uv = \left(\frac{z-y}{2}\right)\left(\frac{z+y}{2}\right) = \left(\frac{x}{2}\right)^2$

Also, we get $\gcd(u,v) = 1$

otherwise if $\gcd(u,v) = d > 1$

then $d \mid (u-v)$ & $d \mid (u+v)$

i.e $d \mid y$ & $d \mid z$

which contradicts $\gcd(y,z) = 1$

$\therefore$ $\gcd(u,v) = 1$

By using lemma (2), we get $u = t^2$ & $v = s^2$

where s & t are positive integers.

Now, substituting all values of u & v

we get $z = v+u = s^2+t^2$

$y = v-u = s^2-t^2$

$x^2 = 4uv = 4s^2t^2$

i.e $x = 2st$.

Now, using values of of z & y & condition

$\gcd(y,z) = 1$

implies $\gcd(s,t) = 1$

if s & t are both even or both odd

which implies y & z are even which

contradicts our fact that taken y & z are odd.

Hence, exactly one of pair s,t is even

and the other is odd (either s is odd or t is even)

i.e $s \not\equiv t \pmod 2$

Conversely, let s & t two integers satisfy given

condition i.e $x = 2st$, $y = s^2-t^2$

$z = s^2+t^2$

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Now, $x^2+y^2 = (2st)^2+(s^2-t^2)^2$

$= (s^2+t^2)^2 = z^2$

we get (x,y,z) form a pythagorean trible.

(To prove) this trible is primitive

i.e $gcd(x,y,z)=1$

let $gcd(x,y,z) = d > 1$

let p (prime) divisor of d

we get $p \neq 2$

if possible $p=2$ As $p|z$ (z is odd integer)

As $z = s^2+t^2$ (one of s & t is odd & other is even)

$\Rightarrow z$ is odd)

And $p|z$ i.e $2|z$ (Not possible)

$\therefore p \neq 2$

From $p|y$ & $p|z$

we obtain $p|(z+y)$ & $p|(z-y)$

i.e $p|2s^2$ & $p|2t^2$ ($\because z+y = s^2+t^2$, $-s^2+t^2 = 2t^2$)

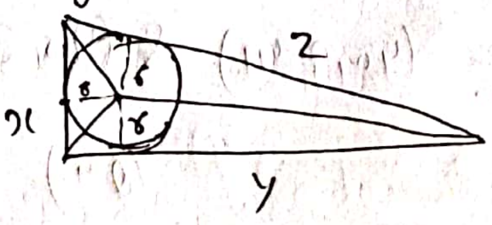
$\Rightarrow p|s$ & $p|t$

which contradicts for $gcd(s,t)=1$

$\therefore$ we get $d=1$

Hence, x,y,z constitutes a primitive Pythagorean Trible. (Proved)

Thm (12.2) (Not proof) The radius of inscribed cycle of a Pythagorean triangle is always an integer.



$r \in \mathbb{Z}$

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Problems 12.1

Q.1 (a) Find three different Pythagorean triples, not necessarily primitive, of the form $16, y, z$.

Sol^n. $x = 16 = 2st$

i.e $s = 8, t = 1$

$s = 4, t = 2$

when $s = 8, t = 1$

$y = s^2 - t^2 = 64 - 1 = 63$

$z = s^2 + t^2 = 64 + 1 = 65$

when $s = 4, t = 2$

$y = s^2 - t^2 = 16 - 4 = 12$

$z = s^2 + t^2 = 16 + 4 = 20$

Thus Pythagorean triples are $(16, 63, 65)$, $(16, 12, 20)$ Ans

Now, let $x = 16 = s^2 - t^2$

$\therefore (s-t)(s+t) = 2 \times 8$

$\therefore s - t = 2, s + t = 8$

$\Rightarrow s = 5, t = 3$

Thus $y = 2st = 2 \times 5 \times 3 = 30$

$z = s^2 + t^2 = 25 + 9 = 34$

Hence pythagorean triple is $(16, 30, 34)$ Ans

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Problem 12.1

Exr. 2 $x+y$ & $x-y$ are congruent modulo 8 as either 1 or 7.

(Sol^n) let us assume that x is even & y is odd then for co-prime $s, t \in N$ s.t $s > t$ & $s \not\equiv t \pmod 2$

(proof (used in thm 12.1)) we have $x = 2st$, $y = s^2 - t^2$

$$\therefore \left.\begin{aligned} x+y &= (s+t)^2 - 2t^2 \\ x-y &= 2t^2 - (s-t)^2 \end{aligned}\right]$$

then, we have $s \pm t$ is odd

Since if s & t both even or both odd which contradicts the fact x, y, z is a primitive pythagorean triple.

$\therefore$ exactly one of pair s, t is even and other is odd. $s > t$ & $\left(\begin{matrix} \text{even}, \text{odd} \\ \text{odd}, \text{even} \end{matrix}\right)$

$$\therefore (s \pm t)^2 \equiv 1 \pmod 8$$

$\therefore$ So if t is even then $x + y \equiv 1 \pmod 8$ (put in ⊛) and $x - y \equiv -1 \equiv 7 \pmod 8$

If t is odd then $x + y \equiv -1 \equiv 7 \pmod 8$

and $x - y \equiv 1 \pmod 8$

(Exr. 6) (Yourself)

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(12.2)

FERMAT'S LAST THEOREM:

Theorem (12.3) (Fermat)
The Diophantine Equation $x^4+y^4=z^2$ has no solution in positive Integer x, y, z

(Corollary) The equation $x^4+y^4=z^4$ has no solution in positive integers.

Theorem (12.4) The Diophantine Equation $x^4-y^4=z^2$ has no solution in positive Integers x, y, z.

Theorem (12.5) The area of a Pythagorean triangle can never be equal to a perfect (integral) square.

Exr. (12.2)
(Q-1) Show that equation $x^2+y^2=z^3$ has infinitely many solutions for x, y, z positive Integers.

Solution) For any $n \geq 2$
let $x = n(n^2-3)$, $y = 3n^2-1$, $z = n^2+1$
then $x^2+y^2 = n^2(n^2-3)^2+(3n^2-1)^2$

$$= n^2(n^4-6n^2+9)+(9n^4-6n^2+1)$$

$$= n^6+3n^4+3n^2+1$$

$$= (n^2+1)^3$$

$$= z^3$$

$\therefore$ there are infinitely many solutions to $x^2+y^2=z^3$.

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