Simultaneous equations of the first order :-

The general type of a set of simultaneous equations of first order between three variables is

$$\left.\begin{aligned} P_1\,dx + Q_1\,dy + R_1\,dz &= 0 \\ P_2\,dx + Q_2\,dy + R_2\,dz &= 0 \end{aligned}\right\} \quad \text{--- (I)}$$

where P_1, Q_1, R_1, P_2, Q_2 & R_2 are functions of x, y, z.

These equations (I) can be expressed in the form

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}} \quad \text{--- (II)}$$

where P, Q & R functions x, y, z.

$$\boxed{\begin{aligned} P &= Q_1R_2 - Q_2R_1 \\ Q &= R_1P_2 - R_2P_1 \\ R &= P_1Q_2 - P_2Q_1 \end{aligned}}$$

Thus eq^n (II) will be taken as the type of a set of simultaneous equations.

We have to find two integral relation by using fractions $\frac{dx}{P}$, $\frac{dy}{Q}$, $\frac{dz}{R}$.

and also we know that the equal fractions $\frac{dx}{P}, \frac{dy}{Q}, \frac{dz}{R}$ are also equal to

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}, \quad \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}, \text{ etc.}$$

Then the system of equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

are all ~~syst~~ satisfied by the same relations between x, y, z that satisfy (II).

Example :- Solve

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{—— (I)}$$

Take the last two fractions of eqn (1),

$$\frac{dy}{2xy} = \frac{dz}{2xz},$$

or

$$\frac{dy}{y} = \frac{dz}{z}$$

By Integrating

$$\log y = \log z + \log c_1$$

or

$$\boxed{y = c_1 z}$$

Now take $l = x$, $m = y$ & $n = z$ as multipliers then

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

Now take last two fractions of this eqn.

then

$$\frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

or

$$\frac{dz}{2z} = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)}$$

or

$$\frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2+y^2+z^2)}$$

By integration

$$\log z = \log(x^2+y^2+z^2) + \log c_2$$

then

$$\log(x^2+y^2+z^2) = \log(c_2 z)$$

or

$$\boxed{x^2+y^2+z^2 = c_2 z}$$ Ans

Then complete soln of given simultaneous eqn is

$$\boxed{\begin{aligned} y &= c_1 z \\ \& \quad x^2+y^2+z^2 &= c_2 z \end{aligned}}$$ Ans

Exercise (1) Solve

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Exercise (4)

(2) Solve $$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

(3) $$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{nxy}$$

(4) Solve $$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2}$$

(5) Solve $$\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy}$$