External Direct Product

Definition. Let $G_1, G_2, ---, G_n$ be a finite collection of groups. The external direct product of $G_1, G_2, ---, G_n$ is written as $G_1 \oplus G_2 \oplus \dots \oplus G_n$ is set of all n-tupes for which the ith component is an element of G_i, and the operation of this is componentwise.

It is denoted by

$$G_1 \oplus G_2 \oplus \dots \oplus G_n = \{(g_1, g_2, ---, g_n) \mid g_i \in G_i\}$$

where $(g_1, g_2, ---, g_n) \in G_1 \oplus G_2 \oplus \dots \oplus G_n$

and $(g_1', g_2', ---, g_n') \in G_1 \oplus G_2 \oplus \dots \oplus G_n$

such that

$$(g_1, g_2, ---, g_n)(g_1', g_2', --, g_n') = (g_1 g_1', g_2 g_2', -- g_n g_n')$$

where operation of $g_i g_i'$ is same as operation of G_i, $1 \le i \le n$

Example $U(8) = \{1, 3, 5, 7\}$, $\otimes_8$

$U(10) = \{1, 3, 7, 9\}$, $\otimes_{10}$

then $U(8) \oplus U(10) = \{(1,1), (1,3), (1,7), (1,9)$
$(3,1), (3,3), (3,7), (3,9)$
$(5,1), (5,3), (5,7), (5,9)$
$(7,1), (7,3), (7,7), (7,9)\}$

consider elements of $U(8) \oplus U(10)$

product $(3,7)(7,9) = (3 \otimes_8 7, 7 \otimes_{10} 9)$
$= (5, 3)$

Example : $Z_2 \oplus Z_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$

consider elements of $Z_2 \oplus Z_3$
find order of element, whether $Z_2 \oplus Z_3$ is Abelian or not ?

(Solution) consider $(1,1)$

$(1,1)^1 = (1,1)$

$(1,1)^2 = (1,1)(1,1) = (1 \oplus_2 1, 1 \oplus_3 1)$
$= (0, 2)$

$(1,1)^3 = (1,1)^2 (1,1) = (0,2)(1,1) = (0 \oplus_2 1, 2 \oplus_3 1)$
$= (1, 0)$

$(1,1)^4 = (1,1)^3 (1,1) = (0,1)$

$(1,1)^5 = (0,1)(1,1) = (1,2)$

$(1,1)^6 = (1,2)(1,1) = (0,0)$

[In External Direct Product $G_1 \oplus G_2$ has identity (e_1, e_2) where e_i is identity of G_i]

$\therefore$ order of $(1,1) = 6$

$\therefore$ $Z_2 \oplus Z_3$ has elt of order 6 and order of $Z_2 \oplus Z_3$ is 6

Also, $|G_1 \oplus G_2| = |G_1| \cdot |G_2|$

$\therefore$ It is cyclic grp

i.e $Z_2 \oplus Z_3$ is cyclic grap of order 6.

$\therefore$ $Z_2 \oplus Z_3 \cong Z_6$ (Using The finite cyclic grap of order $n \cong Z_n$)

Remark: Direct product (external) of abelian grp is abelian

$\therefore$ $Z_2 \oplus Z_3$ is also abelian grp.

Properties of EDP ~~external~~
(External Direct Product)

Theorem: Order of an element in a Direct Product

The order of element of a direct product of a finite number of finite groups is least common multiple of the order of components of elements.

i.e $|(g_1, g_2, \ldots, g_n)| = \text{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$

(proof) let e_i be the identity of G_i.
$G_1 \oplus G_2 \oplus \cdots \oplus G_n$ has identity element $(e_1, e_2, \ldots, e_n)$

let $t = |(g_1, g_2, \ldots, g_n)|$
$s = \text{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$

to show: $t \le s$ and $s \le t$

consider, $(g_1, g_2, \ldots, g_n)$ be arbitrary elt of $G_1 \oplus G_2 \oplus \cdots \oplus G_n$

$(g_1, g_2, \ldots, g_n)^s = (g_1^s, g_2^s, \ldots, g_n^s)$

[since s is multiple of order of g_1
i.e $s = |g_1| k_1$, for some $k_1 \in \mathbb{Z}$
and so on]

$\Rightarrow (g_1, g_2, \ldots, g_n)^s = (g_1^{|g_1| k_1}, g_2^{|g_2| k_2}, \ldots, g_n^{|g_n| k_n})$
$= ((g_1^{|g_1|})^{k_1}, (g_2^{|g_2|})^{k_2}, \ldots, (g_n^{|g_n|})^{k_n})$
$= (e_1^{k_1}, e_2^{k_2}, \ldots, e_n^{k_n})$
$= (e_1, e_2, \ldots, e_n) = e$

$\Rightarrow t$ must divides s
in particular $t \le s$

[using Let G be grp & $a \in G$
be elt of order n.
If $a^k = e$
$\Rightarrow n$ divides k]

Now,
$(g_1^t, g_2^t, \ldots, g_n^t) = (g_1, g_2, \ldots, g_n)^t$
$= (e_1, e_2, \ldots, e_n)$ $\left(\because t = |(g_1, g_2, \ldots, g_n)|\right)$

$\Rightarrow g_1^t = e_1, g_2^t = e_2, \ldots, g_n^t = e_n$
i.e $g_i^t = e_i \quad \forall\, i = 1, 2, \ldots, n$

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$\Rightarrow$ t is multiple of $|g_i|$ for each i

and $s = lcm\ (|g_1|, |g_2|, \ldots, |g_n|)$

$\Rightarrow$ s is smallest multiple of $|g_i|$ for each i

$\Rightarrow$ $s \leq t$

$\therefore$ we get $t = s$

i.e. $|g_1, g_2, \ldots, g_n| = lcm\ (|g_1|, |g_2|, \ldots, |g_n|)$

Application

example 1 $Z_{25} \oplus Z_5$.

Determine the number of elements of order 5 in $Z_{25} \oplus Z_5$.

(Solution) By above result

The number of elements (a, b) in $Z_{25} \oplus Z_5$ of order 5 is given as

$$5 = |(a, b)| = lcm(|a|, |b|)$$

Now, we find number of possibility here, of $|a|, |b|$, so that we get lcm is 5

possibilities, $5 = lcm(|a|, |b|)$

$(1, 5)$

$(5, 1)$

$(5, 5)$

i.e either $|a| = 5$

& $|b| = 1$ or 5

and $|a| = 1$ or 5

& $|b| = 5$

Case ① $(5, 5)$

$|a| = 5$ & $|b| = 5$ ~~$Z_{25} \oplus Z_5$~~ $\left(\begin{array}{l} a \in Z_{25} \\ b \in Z_5 \end{array} \right.$

Now $5 | 25$ in Z_{25}

$\Rightarrow$ number of elements in Z_{25} of order 5 is $\phi(5) = 4$

& $5 | 5$ in Z_5

$\Rightarrow$ number of elements in Z_5 of order 5 is $\phi(5) = 4$

$\therefore$ This gives $4 \times 4 = 16$ elements of order 5

Case ② $(5, 1)$

$|a| = 5$ & $|b| = 1$

Now $5 | 25$ in Z_{25} $\Rightarrow$ number of elts in Z_{25} of order 5 is 4

& $1 | 5$ in Z_5

$\Rightarrow$ number of elts in Z_5 of order 1 is 1 $(\{e\})$

$\Rightarrow$ choices are $4 \times 1 = 4$ elts of order 5

Case ③ $|a| = 1$ & $|b| = 5$

$\Rightarrow$ number of elts in Z_{25} of order 1 is 1

& " " " Z_5 " " 5 is 4

$\Rightarrow$ choices are $1 \times 4 = 4$ elts of order 5

Thus, $Z_{25} \oplus Z_5$ has $16 + 4 + 4 = 24$ elts of order 5.

Example: Determine the number of cyclic subgrps of order 10 in $Z_{100} \oplus Z_{25}$.

(Solution) Cyclic grp are generated by element of order 10
i.e to find number of elements of order 10
in $Z_{100} \oplus Z_{25}$

Firstly, find number of elements of order 10
in $Z_{100} \oplus Z_{25}$

By previous result
$10 = |(a,b)| = lcm(|a|, |b|)$, where $(a,b) \in Z_{10} \oplus Z_{25}$

possibilities are
(1, 10) x
(2, 10) x
(2, 5) ✓
(5, 10) x
(5, 2) x
(10, 1) ✓
(10, 5) ✓
(10, 10) x

Consider (1, 10), $a \in Z_{100}$, $b \in Z_{25}$
and $1 | 100$, $10 \nmid 25$

∴ this case is Not possible

(2, 10), $2 \in Z_{100}$, $10 \in Z_{25}$
and $2 | 100$, $10 \nmid 25$ ⇒ Not possible

Since $10 \nmid 25$
∴ cases (1, 10) X
(2, 10) X
(5, 10) X
(10, 10) X

Also, $2 \nmid 25$

$\Rightarrow (5, 2)$ ✗

Now, $(2, 5)$, $2 \in Z_{100}$, $5 \in Z_{25}$
& $2 \mid 100$, $5 \mid 25$

$\Rightarrow$ number of elts of order 2 in Z_{100} are $\phi(2)$
" " " " 5 " Z_{25} are $\phi(5)$

$\Rightarrow 1 \times 4 = 4$

Now, $(10, 1)$, $10 \mid 100$ & $1 \mid 25$

$\Rightarrow$ number of elts of order 10 in Z_{100} are $\phi(10) = 4$
" " " " 1 " Z_{25} are $\phi(1) = 1$

$\Rightarrow$ ~~...~~ $4 \times 1 = 4$

Now, $(10, 5)$, $10 \mid 100$ & $5 \mid 25$

$\Rightarrow \phi(10) = 4$, $\phi(5) = 4$

$\Rightarrow 4 \times 4 = 16$

$\therefore$ total number of elements of order 10 in $Z_{100} \oplus Z_{25}$ are $16 + 4 + 4 = 24$ elements

Say $\{x_1, x_2, ---, x_{24}\}$ are 24 elements of order 10 in $Z_{100} \oplus Z_{25}$

Then cyclic subgrp generated by x_1 say H_1

i.e $H_1 = \langle x_1 \rangle$

H_1 is cyclic subgrp
& number of generator of $H_1 = \phi(10) = 4$
H_1 has four elts of order 10

Now 4 elts in H_1 subgrp
then 24 elts in subgrp are $\frac{1}{4} \times 24 = 6$

$\therefore$ 6 cyclic subgrp of order 10 in $Z_{100} \oplus Z_{25}$.