1 Classical Statistics

1.1 Macrostate and Microstate, Phase Space, Ensemble, Thermodynamic Probability

Problem 1: Consider a particle undergoing simple harmonic motion such that the position of the particle changes with time as $x = x_0 \cos(\omega t + \phi)$, where the phase ϕ is completely unknown, and therefore the position of the oscillator is not known. One therefore has to resort to determining the probability that the position of the oscillator lies between x and $x + dx$.

(a) This probability must be proportional to the time the oscillator spends between x and $x + dx$. Find the speed of the oscillator at position x as a function of x, ω and x_0. Using this expression, determine the probability $p(x)dx$ that the position of the oscillator is between x and $x + dx$.

(b) Let the energy of the oscillator lie between E and $E + \Delta E$, where $\Delta E << E$. Sketch the phase space and the region accessible to the particle, calculating the volume of the accessible region. Next, compute the ratio of the volume of the accessible phase space corresponding to the position of the particle lying between x and $x + dx$ and the total volume of the accessible phase space. What does this result signify?

Problem 2: Consider an isolated system of four non-interacting spins labelled 1, 2, 3, and 4, each with magnetic moment m, interacting with an external magnetic field B. Each spin can be parallel ('up') or antiparallel ('down') to B, with the energy of a spin parallel to B equal to $\epsilon = -mB$ and the energy of a spin antiparallel to B equal to $\epsilon = +mB$. Let the total energy of the system be $E = -2mB$.

(a) How many microstates of the system correspond to this macrostate? Enumerate these microstates.

(b) What is the probability that the system is in a given microstate in equilibrium?

(c) What is the probability that a given spin points up? Use this probability to campute the mean magnetic moment of a given spin in equilibrium.

(d) What is the probability that if spin 1 is 'up', spin 2 is also 'up'?

Problem 3: Consider a system of four non-interacting distinguishable particles, with each particle localised to a lattice site. The energy of each particle is is restricted to values $\epsilon = 0, \epsilon_0, 2\epsilon_0, 3\epsilon_0, ...$ The system is divided into two subsystems A and B, subsystem A consisting of particles 1 and 2, and B consisting of particles 3 and 4 respectively. A and B are initially thermally insulated from each other, with energies $E_A = 5\epsilon_0$ and $E_B = \epsilon_0$. What are the possible microstates of the composite system? Now, suppose the two subsystems are allowed to thermally interact with each other, so that they can exchange energy without the total energy of the system changing. After equilibrium is attained, enumerate the possible microstates of the composite system. In equilibrium, what is the probability that subsystem A has energy E_A, for $E_A = 0, \epsilon_0, 2\epsilon_0, .., 6\epsilon_0$? For what value of E_A is the probability maximum?

1.2 Entropy and Thermodynamic Probability

Problem 1: Consider a system of N particles (which could be interacting with each other) with energy E and occupying a volume V. The entropy of the system is known to be extensive. Suppose the energy of the system is changed, such that the new energy is λE, where λ is a multiplicative factor. Can you say that the new entropy will be λS, where S is the original entropy? If not, what other changes will be needed such that this is true?

Problem 2: Consider a system of $N >> 1$ weakly interacting particles, each of which can be in quantum states with energies $0, \epsilon, 2\epsilon, 3\epsilon, \ldots$. Given the system has a certain energy, the temperature of the system is given by

$$\frac{1}{T} = \frac{\partial S}{\partial E} \simeq \frac{\Delta S}{\Delta E}$$

where ΔS is the change in the entropy of the system due to the change in the energy of the system by ΔE.

(a) If the system is in its ground state, what is its entropy?

(b) If the total energy of the system is ϵ, what is its entropy?

(c) What is the change in entropy of the system if the total energy of the system is increased from ϵ to 2ϵ?

(d) Given the above definition of temperature, what is the temperature of the system if its total energy is ϵ?

Problem 3: A system of four weakly interacting distinct particles is such that each particle can be in one of four states with energies $\epsilon, 2\epsilon, 3\epsilon$ and 4ϵ respectively. If the system has total energy 15ϵ, what is the entropy of the system? For what possible values of total energy is the entropy of the system zero?

Problem 4: Consider a lattice of N non-interacting distinguishable particles, with each particle localised to a lattice site. The energy of each particle is restricted to values $\epsilon = 0, \epsilon_0, 2\epsilon_0, 3\epsilon_0, \ldots$. The system is in equilibrium.

(a) If the energy of the system is E, what is the number of microstates of the system?

(b) Find an expression for the entropy of the system as a function of energy and simplify it using Sterling's approximation $\ln n \simeq n \ln n - n$ for $n >> 1$.

(c) Using the relation

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

determine a relation between the energy of the system and its temperature.

Hint: The problem of determining the number of microstates can be reduced to counting the number of ways of arranging a certain number of sticks and a certain number of dots along a line.

1.3 Maxwell Boltzmann Distribution

Problem 1: Consider atomic hydrogen in thermal equilibrium at temperature T. Estimate the ratio of the number of atoms with energy $E = -3.4$ eV to the number of atoms with energy $E = -13.6$ eV for $T = 1000°K$.

Problem 2: A system of N weakly interacting particles, each of mass m, is in thermal equilibrium at temperature T. The system is contained in a cubical box of side L, whose top and bottom surfaces are parallel to the Earth's surface, where the acceleration due to gravity is g. A coordinate system is set up with the origin at the centre of the base of the box and the positive z axis along the vertical direction, such that the ranges of coordinates accessible to any particle are $-L/2 \leq x \leq L/2$, $-L/2 \leq y \leq L/2$, $0 \leq z \leq L$.

(a) What is the probability that a given particle has velocity in the range (v_x, v_y, v_z) and $(v_x + dv_x, v_y + dv_y, v_z + dv_z)$?

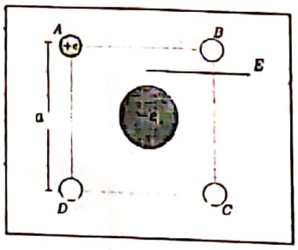
(b) What is the probability that a given particle has x coordinate between x and $x + dx$?

(c) What is the probability that a given particle has y coordinate between y and $y + dy$?

(d) What is the probability that a given particle has z coordinate between z and $z + dz$?

(e) From the above probability distributions, calculate the mean kinetic and potential energies of a particle.

Problem 3: A two-dimensional solid at temperature T contains N negatively charged impurity ions per unit area, the negative ions replacing some ordinary atoms of the solid. The solid as a whole is electrically neutral, since each negative ion with charge $-e$ has in its vicinity one positive ion with charge $+e$. The positive ion, much smaller, is free to move between each of the four equidistance sites A, B, C and D surrounding the stationary negative ion, as shown. The spacing between the these sites is a and the energy of interaction of the positive ion with the stationary negative ion is $-\epsilon_0$ for each lattice site



(a) What are the relative probabilities of the positive ion being found at the four lattice sites?

(b) The solid is placed in a region of a uniform electric field of magnitude E, as illustrated above. Taking the origin at the location of the negative ion, determine the interaction energy of the system with the external electric field at the four lattice sites (*the interaction energy is* $E_{int} = -\vec{p} \cdot \vec{E}$ *where* $\vec{p}$ *is the dipole moment of the system*).

(c) What now are the relative probabilities of the positive ion being found at the four lattice sites?

(d) The mean polarisation of the solid is the mean dipole moment per unit area along the direction of the electric field. Calculate the polarisation of the solid as a function of temperature and the external electric field E.

(e) Calculate the expression for the polarisation at 'high' temperatures. What temperatures are 'high'?

Problem 4: A sensitive spring balance consists of a quartz spring with spring constant k. This balance is used to measure the mass of very tiny, light objects by suspending them from the balance and observing the extension in the spring. Consider a tiny object of mass m suspended from the spring. The object is in an environment which is at temperature T, and gets 'kicked' around by it, reaching equilibrium with the environment.

1. What is the potential energy of the system if the spring is extended by x?
2. What is the probability that the spring is extended by x relative to its equilibrium length?
3. Calculate the mean extension $\overline{x}$ and the mean squared extension $\overline{(x-\overline{x})^2}$.
4. Comparing the square root of the mean squared extension with the mean extension, extimate the minimum mass that can be reliably measured.

1.4 Partition Function