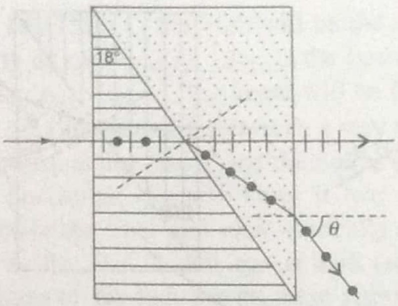

Fig. 22.31 Production of two orthogonally polarized beams by a Rochon prism.

Thus $\sin r = \dfrac{n_o}{n_e} \sin 18°$

$= \dfrac{1.658}{1.486} \times 0.309 \approx 0.345$

$\Rightarrow \quad r = 20.2°$

Therefore the angle of incidence at the second surface will be 20.2 – 18° = 2.2°. The emerging angle will be given by

$\sin \theta = n_e \sin (2.2°) \approx 0.057$

$\Rightarrow \quad \theta \approx 3.3°$

22.12 PLANE WAVE PROPAGATION IN ANISOTROPIC MEDIA

In this section, we will discuss the plane wave solutions of Maxwell's equations in an anisotropic medium and prove the various assumptions made in Sec. 22.5. The difference between an isotropic and an anisotropic medium is in the relationship between the displacement vector **D** and the electric vector **E**; the displacement vector **D** is defined in Sec. 23.9. In an isotropic medium, **D** is in the same direction as **E** and one can write

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{63}$$

where ε is the dielectric permittivity of the medium. On the other hand, in an anisotropic medium **D** is not, in general, in the direction of **E** and the relation between **D** and **E** can be written in the form

$$\left.\begin{aligned} D_x &= \varepsilon_{xx} E_x + \varepsilon_{xy} E_y + \varepsilon_{xz} E_z \\ D_y &= \varepsilon_{yx} E_x + \varepsilon_{yy} E_y + \varepsilon_{yz} E_z \\ D_z &= \varepsilon_{zx} E_x + \varepsilon_{zy} E_y + \varepsilon_{zz} E_z \end{aligned}\right\} \tag{64}$$

where $\varepsilon_{xx}, \varepsilon_{xy}, \ldots$ are constants. One can show that*

$$\varepsilon_{xy} = \varepsilon_{yx}; \quad \varepsilon_{xz} = \varepsilon_{zx}$$

and

$$\varepsilon_{yz} = \varepsilon_{zy} \tag{65}$$

Further, one can always choose a coordinate system (i.e., one can always choose appropriately the directions of x, y and z axes inside the crystal) such that

$$\left.\begin{aligned} D_x &= \varepsilon_x E_x \\ D_y &= \varepsilon_y E_y \\ \text{and} \quad D_z &= \varepsilon_z E_z \end{aligned}\right\} \tag{66}$$

This coordinate system is known as the principal axis system and the quantities ε_x, ε_y and ε_z are known as the principle dielectric permittivities of the medium. If

$$\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z \quad \text{(biaxial)} \tag{67}$$

we have what is known as a biaxial medium and the quantities

$$n_x = \sqrt{\frac{\varepsilon_x}{\varepsilon_0}}, \; n_y = \sqrt{\frac{\varepsilon_y}{\varepsilon_0}}, \; n_z = \sqrt{\frac{\varepsilon_z}{\varepsilon_0}} \tag{68}$$

are said to be the principal refractive indices of the medium; in the above equation ε_0 represents the dielectric permittivity of free space (= 8.8542×10^{-12} C^2/N-m^2). If

$$\varepsilon_x = \varepsilon_y \neq \varepsilon_z \quad \text{(uniaxial)} \tag{69}$$

we have what is known as a uniaxial medium with the z-axis representing the optic axis of the medium. The quantities

$$n_o = \sqrt{\frac{\varepsilon_x}{\varepsilon_0}} = \sqrt{\frac{\varepsilon_y}{\varepsilon_0}}$$

and

$$n_e = n_z = \sqrt{\frac{\varepsilon_z}{\varepsilon_0}} \tag{70}$$

are known as ordinary and extra-ordinary refractive indices; typical values for some uniaxial crystals are given in Table 22.1. For a uniaxial medium, since $\varepsilon_x = \varepsilon_y$ the x and y directions can be arbitrarily chosen as long as they are perpendicular to the optic axis, i.e., *any* two mutually perpendicular axes (which are also perpendicular to the z-axis) can be taken as the principal axes of the medium.** On the other hand, if

$$\varepsilon_x = \varepsilon_y = \varepsilon_z \quad \text{(isotropic)} \tag{71}$$

---

* See, e.g., Ref. 5.

** This follows from the fact that for a uniaxial medium

$$D_x = \varepsilon_x E_x \text{ and } D_y = \varepsilon_y E_y = \varepsilon_x E_y$$

Now, if we rotate the x-y axes (about the z-axis) by an angle θ and call the rotated axes x' and y', then

$$D_{x'} = D_x \cos\theta + D_y \sin\theta = \varepsilon_x [E_x \cos\theta + E_y \sin\theta]$$
$$= \varepsilon_x E_{x'}$$

Similarly $D_{y'} = \varepsilon_x E_{y'}$, implying that the x'-y' axes ...

Table 22.1 Ordinary and extra-ordinary refractive Indices for some uniaxial crystals (Table adapted from Ref. 6 and 7).

Name of the crystal	Wavelength	n_o	n_e
Calcite	4046 Å	1.68134	1.49694
	5890 Å	1.65835	1.48640
	7065 Å	1.65207	1.48359
Quartz	5890 Å	1.54424	1.55335
Lithium niobate	6000 Å	2.2967	2.2082
KDP	6328 Å	1.50737	1.46685
ADP	6328 Å	1.52166	1.47685

we have an isotropic medium, and can choose any three mutually perpendicular axes as the principal axis system. We will assume the anisotropic medium to be non-magnetic so that

$$\mathbf{B} = \mu_0 \mathbf{H}$$

where μ_0 is the free space magnetic permeability.

Let us consider the propagation of a plane electromagnetic wave; for such a wave the vectors **E**, **H**, **D** and **B** would be proportional to exp $[i(\mathbf{k}\,.\,\mathbf{r} - \omega t)]$. Thus

$$\left.\begin{array}{ll} \mathbf{E} = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} & \mathbf{H} = \mathbf{H}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ \mathbf{D} = \mathbf{D}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} & \mathbf{B} = \mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \end{array}\right\} \tag{72}$$

where the vectors $\mathbf{E}_0$, $\mathbf{H}_0$, $\mathbf{D}_0$ and $\mathbf{B}_0$ are independent of space and time; **k** represents the propagation vector of the wave and ω the angular frequency. The wave velocity v_w (also known as the phase velocity) and the wave refractive index n_w are defined through the following equation:

$$v_w = \frac{\omega}{k} = \frac{c}{n_w} \tag{73}$$

Thus

$$|\mathbf{k}| = k = \frac{\omega}{c} n_w \tag{74}$$

In the present section, it is our objective to determine the possible values of n_w when a plane wave propagates through an anisotropic dielectric. Now, in a dielectric medium

$$\text{div}\, \mathbf{D} = 0 \tag{75}$$

or

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0$$

For a plane wave given by Eq. (72) the above equation becomes

$$i\,(k_x D_x + k_y D_y + k_z D_z) = 0$$

or

$$\mathbf{D}\,.\,\mathbf{k} = 0 \tag{76}$$

implying that **D** is *always* at right angles to **k** [see Eq. (44)]. Similarly since in a non-magnetic medium div **H** = 0,

H *will always be right angles to* **k**. (77)

Now, in the absence of any currents (i.e, **J** = 0) Maxwell's curl equations [see Eqs (7) and (8) of Chapter 23] become

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = i\omega\mathbf{B} = i\omega\mu_0\mathbf{H} \tag{78}$$

and

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = -\, i\omega\mathbf{D} \tag{79}$$

where we have assumed the medium to be non-magnetic (i.e., $\mathbf{B} = \mu_0\, \mathbf{H}$). Now, if

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\mathbf{k}.\mathbf{r} - \omega t)}$$

then

$$(\nabla \times \mathbf{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}$$

$$= (ik_y E_{0z} - ik_z E_{0y})\, e^{i(\mathbf{k}.\mathbf{r} - \omega t)}$$

$$= i(k_y E_z - k_z E_y) = i(\mathbf{k} \times \mathbf{E})_x$$

Thus

$$\nabla \times \mathbf{E} = i\,(\mathbf{k} \times \mathbf{E}) = i\omega\mu_0\, \mathbf{H}$$

$$\Rightarrow \quad \mathbf{H} = \frac{1}{\omega\mu_0}\,(\mathbf{k} \times \mathbf{E}) \tag{80}$$

and

$$\nabla \times \mathbf{H} = i(\mathbf{k} \times \mathbf{H}) = -\, i\omega\mathbf{D}$$

$$\Rightarrow \quad \mathbf{D} = \frac{1}{\omega}\,(\mathbf{H} \times \mathbf{k}) \tag{81}$$

Equations (80) and (81) show that

H *is at right angles to* **k**, **E** *and* **D** (82)

implying

k, **E** *and* **D** *will always be in the same plane.*

Further [see Eq. (76)]

D *is at right angles to* **k** (83)

Substituting for **H** in Eq. (81), we get

$$\mathbf{D} = \frac{1}{\omega^2 \mu_0}\,[(\mathbf{k}\,.\,\mathbf{k})\,\mathbf{E} - (\mathbf{k}\,.\,\mathbf{E})\,\mathbf{k}] \tag{84}$$

where we have used the vector identity

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A}\,.\,\mathbf{C})\,\mathbf{B} - (\mathbf{B}\,.\,\mathbf{C})\,\mathbf{A}$$

Thus

$$\mathbf{D} = \frac{k^2}{\omega^2 \mu_0}\,[\mathbf{E} - (\hat{\kappa}\,.\,\mathbf{E})\,\hat{\kappa}]$$

$$= \frac{n_w^2}{c^2 \mu_0}\,[\mathbf{E} - (\hat{\kappa}\,.\,\mathbf{E})\,\hat{\kappa}] \tag{85}$$

where

$$\hat{\kappa} = \frac{\mathbf{k}}{k} \quad (86)$$

represents the unit vector along **k** (see Fig. 22.32). Since

$$D_x = \varepsilon_x E_x = \varepsilon_0 n_x^2 E_x$$

we have for the *x*-component of Eq. (85)

$$\frac{\varepsilon_0 \mu_0 c^2 n_x^2}{n_w^2} E_x = E_x - \kappa_x (\kappa_x E_x + \kappa_y E_y + \kappa_z E_z)$$

Since $c^2 = 1/(\varepsilon_0 \mu_0)$, we have

$$\left(\frac{n_x^2}{n_w^2} - \kappa_y^2 - \kappa_z^2\right) E_x + \kappa_x \kappa_y E_y + \kappa_x \kappa_z E_z = 0 \quad (87)$$

where we have used the relation $\kappa_x^2 + \kappa_y^2 + \kappa_z^2 = 1$ (since $\hat{\kappa}$ is a unit vector). Similarly,

$$\kappa_x \kappa_y E_x + \left(\frac{n_y^2}{n_w^2} - \kappa_x^2 - \kappa_z^2\right) E_y + \kappa_y \kappa_z E_z = 0 \quad (88)$$

$$\kappa_x \kappa_z E_x + \kappa_y \kappa_z E_y + \left(\frac{n_z^2}{n_w^2} - \kappa_x^2 - \kappa_y^2\right) E_z = 0 \quad (89)$$

Since the above equations form a set of three homogenous equations, for non-trivial solutions, we must have

$$\begin{vmatrix} \frac{n_x^2}{n_w^2} - \kappa_y^2 - \kappa_z^2 & \kappa_x \kappa_y & \kappa_x \kappa_z \\ \kappa_x \kappa_y & \frac{n_y^2}{n_w^2} - \kappa_x^2 - \kappa_z^2 & \kappa_y \kappa_z \\ \kappa_x \kappa_z & \kappa_y \kappa_z & \frac{n_z^2}{n_w^2} - \kappa_x^2 - \kappa_y^2 \end{vmatrix} = 0 \quad (90)$$

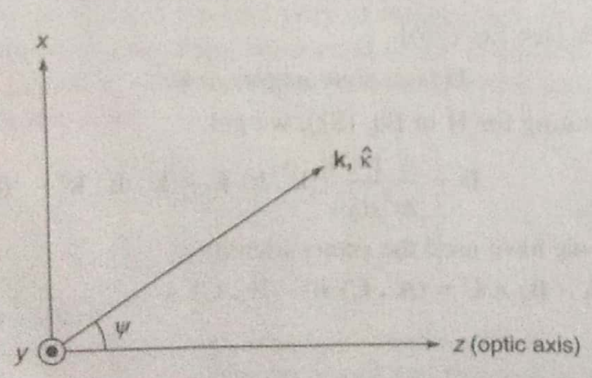

Fig. 22.32 In uniaxial crystals, we can always choose the *y* axis in such a way that $\kappa_y = 0$; the optic axis is assumed to be in the *z*-direction. If ψ is the angle that k makes with the optic axis then $\kappa_x = \kappa \sin \psi$ and $\kappa_z = \kappa \cos \psi$.

We should remember that we still do not know the possible values of n_w. Indeed, for a given direction of propagation (i.e., for given values of κ_x, κ_y and κ_z) the solutions of Eq. (90) gives us the two allowed values of n_w. It may be mentioned that from Eq. (90) it appears as if we will have a cubic equation in n_w^2 which would give us three roots of n_w^2; however, the coefficient of n_w^6 will always be zero and hence there will be *always* two roots. We illustrate the general procedure by considering propagation through a uniaxial medium.

22.12.1 Propagation in Uniaxial Crystals

In this section, we will completely restrict ourselves to uniaxial crystals for which

$$n_x = n_y = n_o \quad \text{and} \quad n_z = n_e \quad (91)$$

As discussed earlier, for a uniaxial crystal, the *x* and *y* directions can be arbitrarily chosen as long as they are perpendicular to the optic axis. Now, for a wave propagating along *any* direction **k**, we choose our *y*-axis in such a way that it is at right angles to **k**, i.e., the *y*-axis is normal to the plane defined by **k** and the *z*-axis; obviously, the *x*-axis will lie in the same plane (see Fig. 22.32). Thus we may write

$$\kappa_x = \sin \psi, \quad \kappa_y = 0$$

and

$$\kappa_z = \cos \psi$$

where ψ is the angle that the **k** vector makes with the optic axis (see Fig. 22.32). Equations (87)–(89) therefore become

$$\left(\frac{n_o^2}{n_w^2} - \cos^2 \psi\right) E_x + \sin \psi \cos \psi \, E_z = 0 \quad (92)$$

$$\left(\frac{n_o^2}{n_w^2} - 1\right) E_y = 0 \quad (93)$$

and

$$\sin \psi \cos \psi \, E_x + \left(\frac{n_e^2}{n_w^2} - \sin^2 \psi\right) E_z = 0 \quad (94)$$

Once again we have a set of three homogenous equations and for non-trivial solutions, the determinant must be zero. However, since two equations involve only E_x and E_z and one equation involves only E_y we have the following two independent solutions:

First Solution: We assume $E_y \neq 0$ then $E_x = 0 = E_z$. From Eq. (93) one obtains the solution

$$n_w = n_{wo} = n_o \quad \text{(ordinary wave)} \quad (95)$$

The corresponding wave velocity is

$$v_w = v_{wo} = \frac{c}{n_o} \quad (y\text{-polarized } o\text{-wave}) \qquad (96)$$

Since the wave velocity is independent of the direction of the wave, it is referred to as the ordinary wave (usually abbreviated as the *o*-wave) and hence the subscript '*o*' on n_w and v_w. Further, for the *o*-wave, the **D** vector (and the **E** vector) is *y*-polarized. Thus, *for the o-wave, the* **D** *vector (and the* **E** *vector) are perpendicular to the plane containing the* **k** *vector and the optic axis* (see Fig. 22.33). This was the recipe that was given through Eq. (43).

Second Solution: The second solution of Eqs (92) – (94) will correspond to

$$E_y = 0;\ E_x,\ E_z \neq 0 \qquad (97)$$

We use Eqs (92) – (94) to obtain

$$\frac{E_z}{E_x} = -\frac{\dfrac{n_o^2}{n_w^2} - \cos^2\psi}{\sin\psi\cos\psi} = -\frac{\sin\psi\cos\psi}{\dfrac{n_e^2}{n_w^2} - \sin^2\psi}$$

Simple manipulations would give us

$$\frac{1}{n_w^2} = \frac{1}{n_{we}^2} = \frac{\cos^2\psi}{n_o^2} + \frac{\sin^2\psi}{n_e^2} \qquad (98)$$

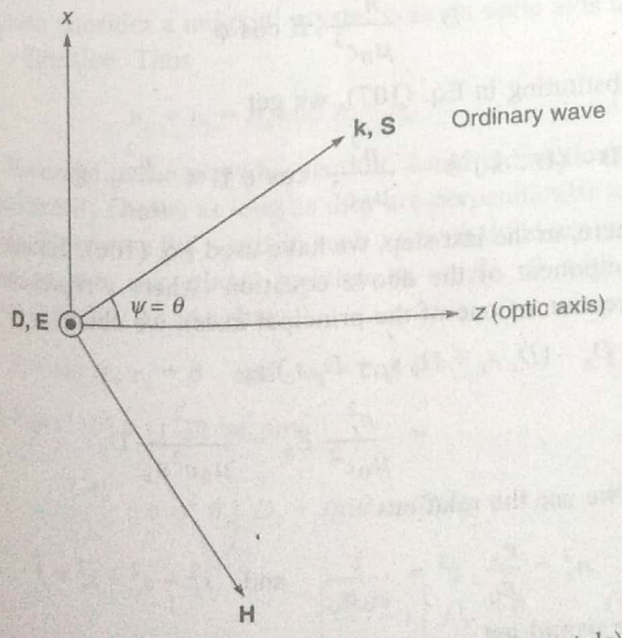

Fig. 22.33 For the ordinary wave (in uniaxial crystals), **D** and **E** vectors are in the *y* direction; **k** and **S** are in the same direction in the *x*-*z* plane and **H** also lies in the *x*-*z* plane.

where the subscript *e* refers to the fact that the wave refractive index corresponds to the extra-ordinary wave. The corresponding wave velocity would be given by

$$v_{we}^2 = \frac{c^2}{n_{we}^2} = \frac{c^2}{n_o^2}\cos^2\psi + \frac{c^2}{n_e^2}\sin^2\psi \qquad (99)$$

Since the wave velocity is dependent on the direction of the wave, it is referred to as the extra-ordinary wave and hence the subscript *e*. Of course, for the extra-ordinary wave, we must have

$$D_y = \varepsilon_y E_y = 0$$

From the above equation and Eq.(81), it follows that the displacement vector **D** of the wave is normal to the *y*-axis and also to **k** implying that *the displacement vector* **D** *associated with the extraordinary wave lies in the plane containing the propagation vector* **k** *and the optic axis and is normal to* **k** (see Fig. 22.34). This was the recipe given through Eq. (44). Figure 22.34 also shows the Poynting vector **S** (= **E** × **H**) which represents the direction of energy propagation (i.e., the direction of the *e*-ray). The small dashes on the extraordinary ray in Figs 22.21(a) and (b) represent the directions of the **D** vector. Let ϕ and θ represent the angles that the **S** vector makes with the **k** vector and the optic axis respectively (see Fig. 22.33). In order to determine the angle ϕ we note that

$$\frac{\varepsilon_z E_z}{\varepsilon_x E_x} = \frac{D_z}{D_x} = -\tan\psi$$

and since

$$\frac{E_z}{E_x} = -\tan(\phi + \psi) \qquad (100)$$

we get

$$\frac{n_e^2}{n_o^2}\tan(\phi + \psi) = \tan\psi$$

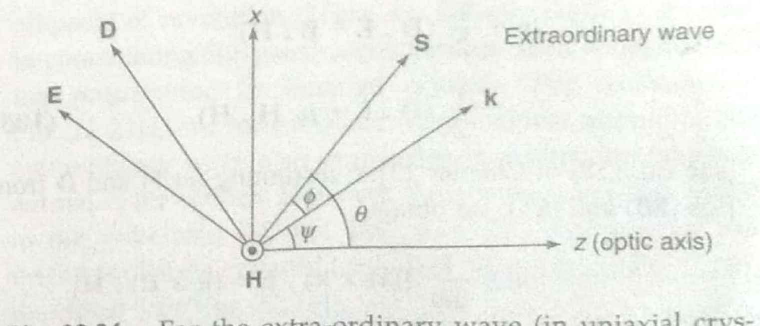

Fig. 22.34 For the extra-ordinary wave (in uniaxial crystals), **E**, **D**, **S** and **k** vectors would lie in the *x* – *z* plane and **H** will be in the *y* direction. **S** is at right angles to **E** and **H**; **D** is at right angles to **k** and **H**.

or,

$$\phi = \tan^{-1}\left[\frac{n_o^2}{n_e^2}\tan\psi\right] - \psi$$

Obviously, for negative crystals $n_o > n_e$ and ϕ will be positive implying that ray direction is further away from the optic axis as shown in Fig. 22.34.

Conversely, for positive crystals $n_o < n_e$ and ϕ will be negative implying that the ray direction will be towards the optic axis.

Example 22.3 We consider calcite for which (at $\lambda = 5893$ Å and 18°C)

$$n_o = 1.65835,\ n_e = 1.48640$$

If we consider **k** making an angle of 30° to the optic axis, then ψ = 30° and elementary calculations give us $\phi = 5.7°$

22.13 RAY VELOCITY AND RAY REFRACTIVE INDEX

The direction of energy propagation (or the ray propagation) is along the Poynting vector **S** which is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad (101)$$

Thus, since the plane containing the vectors **k**, **E** and **D** is normal to **H**, the Poynting vector **S** will also lie in the plane containing the vectors **k**, **E** and **D** (see Figs 22.33 and 22.34). For the extra-ordinary wave, the direction of the propagation of the wave ($\hat{\kappa}$) is not along the direction of energy propagation ($\hat{s}$), where $\hat{s}$ is the unit vector along **S**. The ray velocity (or the energy transmission velocity) v_r is defined as

$$v_r = \frac{S}{u} \qquad (102)$$

where u is the energy density. Now,

$$u = \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H})$$

$$= \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mu_0\mathbf{H}\cdot\mathbf{H}) \qquad (103)$$

[see Eq. (58) of Chapter 23]. Substituting for **H** and **D** from Eqs (80) and (81), we obtain

$$u = \frac{1}{2\omega}[(\mathbf{H}\times\mathbf{k})\cdot\mathbf{E} + (\mathbf{k}\times\mathbf{E})\cdot\mathbf{H}]$$

$$= \frac{1}{2\omega}[\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{k}\cdot(\mathbf{E}\times\mathbf{H})]$$

$$= \frac{1}{\omega}\mathbf{k}\cdot\mathbf{S} \qquad (104)$$

Thus Eq. (102) becomes

$$v_r = \frac{\omega S}{\mathbf{k}\cdot\mathbf{S}} = \frac{\omega}{k\cos\phi} = \frac{v_w}{\cos\phi} \qquad (105)$$

where ϕ is the angle between $\hat{\kappa}$ and $\hat{s}$ (see Fig. 22.34). The ray refractive index n_r is defined as

$$n_r = \frac{c}{v_r} = \frac{c}{v_w}\cos\phi = n_w\cos\phi \qquad (106)$$

In order to express **E** in terms of **D**, we refer to Fig. 22.34 and write

$$\mathbf{D} = (\mathbf{D}\cdot\hat{\mathbf{e}})\,\hat{\mathbf{e}} + (\mathbf{D}\cdot\hat{\mathbf{s}})\,\hat{\mathbf{s}}$$

wher $\hat{\mathbf{e}}$ is a unit vector along the direction of the electric field **E**. Thus

$$\mathbf{D} - (\mathbf{D}\cdot\hat{\mathbf{s}})\,\hat{\mathbf{s}} = (\mathbf{D}\cdot\hat{\mathbf{e}})\,\hat{\mathbf{e}} = (D\cos\phi)\frac{\mathbf{E}}{E} \qquad (107)$$

Similarly,

$$\mathbf{E} = (\mathbf{E}\cdot\hat{\mathbf{d}})\,\hat{\mathbf{d}} + (\mathbf{E}\cdot\hat{\kappa})\,\hat{\kappa} \qquad (108)$$

where $\hat{\mathbf{d}}$ represents a unit vector along the displacement vector **D** (see Fig. 22.34). If we now substitute for $\mathbf{E} - (\mathbf{E}\cdot\hat{\kappa})\,\hat{\kappa}$ in Eq. (85), we would get

$$\mathbf{D} = \frac{n_w^2}{\mu_0 c^2}(\mathbf{E}\cdot\hat{\mathbf{d}})\,\hat{\mathbf{d}}$$

or

$$D = \frac{n_w^2}{\mu_0 c^2}E\cos\phi \qquad (109)$$

Substituting in Eq. (107), we get

$$\mathbf{D} - (\mathbf{D}\cdot\hat{\mathbf{s}})\,\hat{\mathbf{s}} = \frac{n_w^2}{\mu_0 c^2}\cos^2\phi\,\mathbf{E} = \frac{n_r^2}{\mu_0 c^2}\mathbf{E}$$

where, in the last step, we have used Eq. (106). Taking the x component of the above equation (where x represents the direction of one of the principal axes), we obtain

$$D_x - (D_x s_x + D_y s_y + D_z s_z)\,s_x$$

$$= \frac{n_r^2}{\mu_0 c^2}E_x = \frac{n_r^2}{\mu_0 c^2 \varepsilon_x}D_x$$

If we use the relations

$$n_x^2 = \frac{\varepsilon_x}{\varepsilon_0},\ c^2 = \frac{1}{\varepsilon_0\mu_0} \quad \text{and} \quad s_x^2 + s_y^2 + s_z^2 = 1$$

we would get

$$\left(\frac{n_r^2}{n_x^2} - s_y^2 - s_z^2\right)D_x + s_x s_y D_y + s_x s_z D_z = 0 \qquad (110)$$

Similarly

$$s_x s_y D_x + \left(\frac{n_r^2}{n_y^2} - s_x^2 - s_z^2 \right) D_y + s_z s_y D_z = 0 \qquad (111)$$

$$s_x s_z D_x + s_z s_y D_y + \left(\frac{n_r^2}{n_z^2} - s_x^2 - s_y^2 \right) D_z = 0 \qquad (112)$$

As in the previous section, the above set of equations form a set of three homogenous equations. For non-trivial solutions, we must have

$$\begin{vmatrix} \frac{n_r^2}{n_x^2} - s_y^2 - s_z^2 & s_x s_y & s_x s_z \\ s_x s_y & \frac{n_r^2}{n_y^2} - s_x^2 - s_z^2 & s_z s_y \\ s_x s_z & s_z s_y & \frac{n_r^2}{n_z^2} - s_x^2 - s_y^2 \end{vmatrix} = 0 \qquad (113)$$

We still do not know the possible values of n_r. Indeed for a given ray direction (i.e., for given values of s_x, s_y and s_z) the solution of the above equation gives us the two allowed values of n_r and hence two possible values of the ray velocities. We illustrate this by considering propagation through uniaxial media.

22.13.1 Ray Propagation in Uniaxial Crystals

We next consider a uniaxial crystal with its optic axis along the z-direction. Thus

$$n_x = n_y = n_o \text{ and } n_z = n_e \qquad (114)$$

As discussed in the previous section, x and y directions can be arbitrarily chosen as long as they are perpendicular to the z-axis. We choose the y-axis in such a way that the ray propagates in the x–z plane making an angle θ with the z-axis (see Fig. 22.34); thus

$$s_x = \sin\theta,\ s_y = 0 \quad \text{and} \quad s_z = \cos\theta \qquad (115)$$

and Eqs (110) – (112) become

$$\left(\frac{n_r^2}{n_o^2} - \cos^2\theta \right) D_x + \sin\theta\cos\theta\, D_z = 0 \qquad (116)$$

$$\left(\frac{n_r^2}{n_o^2} - 1 \right) D_y = 0 \qquad (117)$$

$$\sin\theta\cos\theta\, D_x + \left(\frac{n_r^2}{n_e^2} - \sin^2\theta \right) D_z = 0 \qquad (118)$$

Obviously, one of the roots is given by

$$n_r = n_{ro} = n_o$$

with

$$D_x = 0 = D_z \text{ (}y\text{-polarized)} \qquad (119)$$

The corresponding ray velocity is given by

$$v_r = v_{ro} = \frac{c}{n_{ro}} = \frac{c}{n_o} \quad \text{(ordinary ray)} \qquad (120)$$

Since the ray velocity is independent of the direction of the ray, it is referred to as the ordinary ray and hence the subscript 'o' on v_r and n_r.

In order to obtain the other solution we use Eqs (116) and (118) to obtain

$$\frac{D_z}{D_x} = -\frac{\frac{n_r^2}{n_o^2} - \cos^2\theta}{\sin\theta\cos\theta} = -\frac{\sin\theta\cos\theta}{\frac{n_r^2}{n_o^2} - \sin^2\theta}$$

and obviously,

$$D_y = 0$$

Simple manipulations give us

$$n_r^2 = n_{re}^2 = n_o^2\cos^2\theta + n_e^2\sin^2\theta \quad \text{(extra-ordinary ray)} \qquad (121)$$

with

$$\frac{D_z/n_e^2}{D_x/n_o^2} = \frac{E_z}{E_x} = -\tan\theta,\ (D_y = 0) \qquad (122)$$

The corresponding ray velocity is given by [cf Eq. (37)]

$$\frac{1}{v_r^2} = \frac{1}{v_{re}^2} = \frac{n_{re}^2}{c^2} = \frac{\cos^2\theta}{c^2/n_o^2} + \frac{\sin^2\theta}{c^2/n_e^2} \qquad (123)$$

which corresponds to the extra-ordinary ray and hence the subscript 'e' on v_r and n_r. As discussed in Sec. 22.5, the above equation represents an ellipse and if we rotate it around the z-axis (i.e., the optic axis) we will get an ellipsoid of revolution. These *ray velocity surfaces* are used in constructing Huygens' secondary wavelets while discussing propagation in uniaxial crystals. For example, in Fig. 22.21(a) we have a plane wave incident normally. The extraordinary wave also propagates in a direction which is normal to the surface. However, the extraordinary rays travel in the directions BE and DE' with EE' representing the wavefront for the extraordinary wave. Returning to Eq. (120), we obtain [see Fig. 22.34]:

$$\tan\theta = -\frac{D_z/n_e^2}{D_x/n_o^2} = \frac{n_o^2}{n_e^2}\tan\psi \qquad (124)$$

Thus when the wave propagates along a direction which makes an angle ψ with the optic axis, then the ray will propagate along the direction

$$\theta = \tan^{-1}\left[\frac{n_o^2}{n_e^2}\tan\psi\right] \tag{125}$$

As an example, for calcite

$$n_o = 1.65836,\ n_e = 1.48641 \text{ with } \psi = 30^\circ$$

we obtain $\theta \approx 35.7^\circ$. Thus the ray direction is further away from the optic axis, consistent with what is shown in Fig. 22.21. It may be noted that $\theta = \phi + \psi$ (see Example 22.3).

22.14 JONES CALCULUS

Through Jones calculus, it becomes quite straightforward to determine the polarization state of the beam emerging from a polarizer or a phase retarder (like a QWP or a HWP). We will illustrate this through some simple examples. We use the exponential notation—for example, a y-polarized beam (propagating in the x-direction) is described by

$$\mathbf{E} = \hat{y}\, E_0 \cos(k\, x - \omega t) = \hat{y}\ \mathrm{Re}\,[E_0\, e^{i(kx-\omega t)}] \tag{126}$$

Such a wave is represented by the vector

$$|\,y - \text{polarized}\,\rangle = \begin{pmatrix}1\\0\end{pmatrix} E_0 \tag{127}$$

Similarly, a z-polarized wave is given by

$$|\,z - \text{polarized}\,\rangle = \begin{pmatrix}0\\1\end{pmatrix} E_0 \tag{128}$$

Now, for an RCP (propagating in the x-direction) we may write

$$E_y = E_0 \cos(kx - \omega t)$$
$$E_z = -E_0 \sin(kx - \omega t)$$

which in the exponential notation can be written as

$$\mathbf{E} = \hat{y}\, e^{i(kx-\omega t)} + \hat{z}\, e^{i(kx-\omega t + \pi/2)}$$

Thus neglecting the phase factor,

$$|\,\text{RCP}\,\rangle = \begin{pmatrix}1\\ e^{i\pi/2}\end{pmatrix} E_0 = \begin{pmatrix}1\\ i\end{pmatrix} E_0 \tag{129}$$

Similarly

$$|\,\text{LCP}\,\rangle = \begin{pmatrix}1\\ -i\end{pmatrix} E_0 \tag{130}$$

Let us next consider a phase retarder like a QWP or a HWP or even an elliptic core fiber. As discussed in earlier sections, the 'modes' of such a device are linearly polarized along the fast and slow axes as shown in Fig.22.24. The electric fields along these directions are denoted by E_f and E_s; the subscripts f and s denote the fast and slow axes respectively. As an example, we consider a calcite QWP for which $n_o > n_e$. The extra-ordinary wave is z-polarized (i.e., along the optic axis) and its velocity ($= c/n_e$) is more than the velocity of the o-wave ($= c/n_o$). Thus (for calcite) the *fast-axis* is along the z-direction and the *slow-axis* is along the y-direction as shown in Fig. 22.24.

The slow and fast components are the *modes* of the retardation plate; i.e., after propagating through the retardation plates (of thickness d), the fields would be given by

$$E_s' = e^{ik_s d}\, E_s$$
$$E_f' = e^{ik_f d}\, E_f$$

where

$$k_s = \frac{2\pi}{\lambda_0} n_s \quad \text{and} \quad k_f = \frac{2\pi}{\lambda_0} n_f$$

(For calcite $n_s = n_o = 1.65836$ and $n_f = n_e = 1.48641$ at $\lambda_0 = 5893$ Å). Since, only the relative phase difference is of interest, we may write

$$E_s' = e^{i\Phi}\, E_s$$
$$E_f' = E_f$$

where

$$\Phi = \frac{2\pi}{\lambda_0}(n_s - n_f)\, d$$

is the phase difference introduced by the phase retarder. The calcite plate is therefore represented by the following matrix

$$\begin{pmatrix} e^{i\Phi} & 0\\ 0 & 1\end{pmatrix}$$

Thus we may write

$$\begin{pmatrix}E_s'\\E_f'\end{pmatrix} = \begin{pmatrix} e^{i\Phi} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}E_s\\E_f\end{pmatrix}$$

As mentioned earlier, y-and z-axes are the slow and fast axes respectively. Thus

$$\begin{pmatrix}E_y'\\E_z'\end{pmatrix} = \begin{pmatrix} e^{i\Phi} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}E_y\\E_z\end{pmatrix}$$

For a y-polarized wave

$$E_y = E_0,\ E_z = 0$$

and a y-polari...