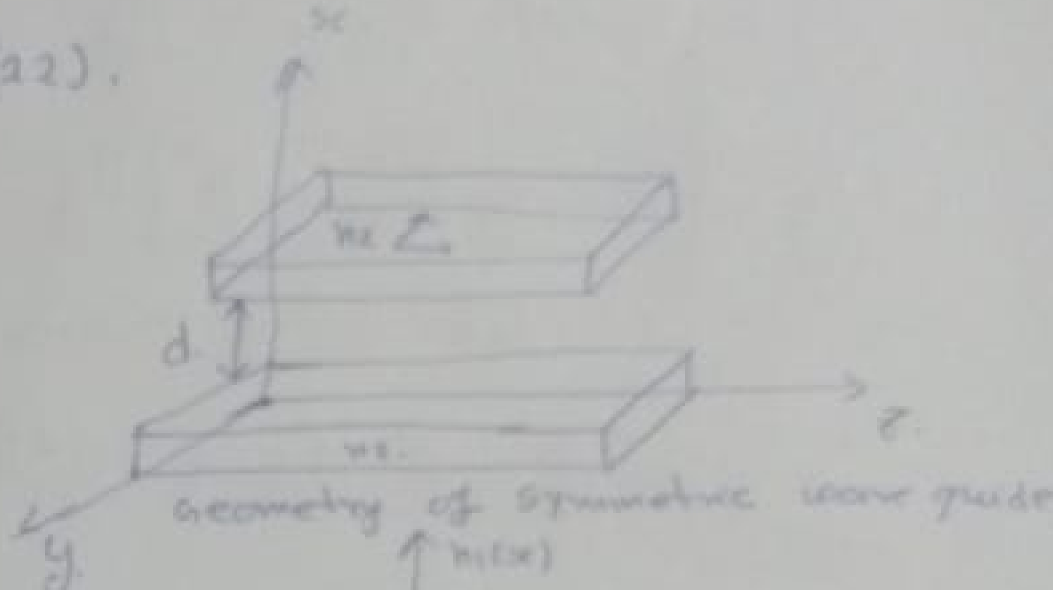
TE modes: symmetric step Index Planar waveguide.
$[E_z = 0]$.

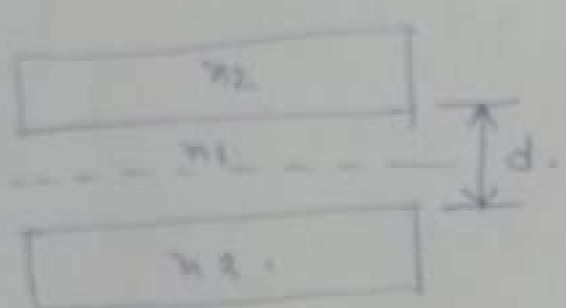
From part- I.

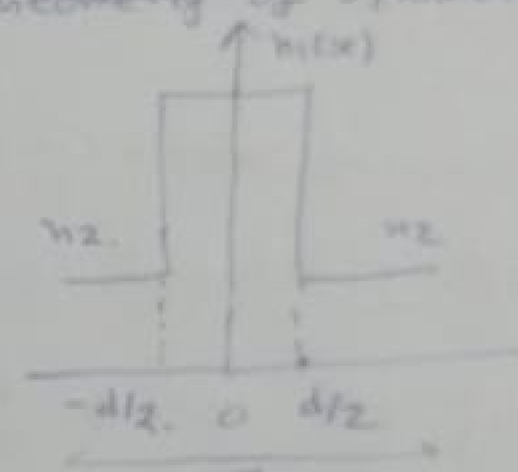
$$H_x = -\frac{\beta}{\omega\mu_0} E_y \text{ - eq. (20).}$$

$$H_z = \frac{i}{\omega\mu_0} \frac{\partial E_y}{\partial x} \text{ from eq. (22).}$$

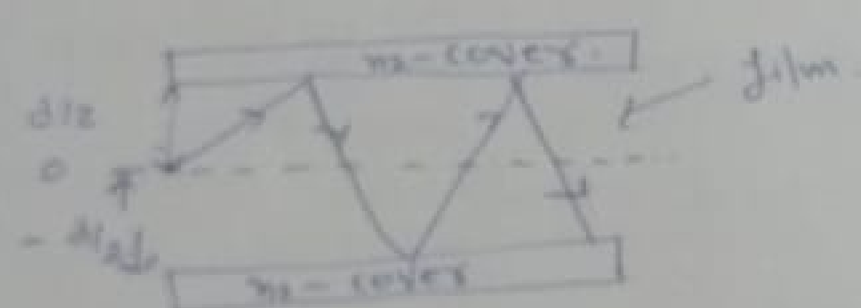


Geometry of symmetric wave guide.





Index profile.



$n = n^2(x)$ i.e. x dependent.

$$n(x) = \begin{cases} n_1 & |x| < d/2 \text{ (film)} \\ n_2 & |x| > d/2 \text{ (cover)} \end{cases} \quad (23)$$

$n_1 > n_2$. This type of index profile is usually referred to as a step-index profile. The wave guide is assumed to extend to infinity in y and z directions.

T.E mode: Substitute for H_x (eq. 20) and H_z eq(22) in equation. (24), ie in

$$-\frac{\partial H_z}{\partial x} - i\beta H_x = i\omega\epsilon_0 n^2(x) E_y. \text{ — (24)}$$

$$-\frac{\partial}{\partial x}\left[\cancel{-\frac{\beta}{\omega\mu_0} E_y}\right] - i\beta\left[-\frac{\beta}{\omega\mu_0} E_y\right] = i\omega\epsilon_0 n^2(x) E_y.$$

$$-\left[\frac{\partial}{\partial x}\left(\frac{i}{\omega\mu_0}\frac{\partial E_y}{\partial x}\right)\right] - i\beta\left[-\frac{\beta}{\omega\mu_0} E_y\right] = i\omega\epsilon_0 n^2(x) E_y$$

$$\frac{E_y \beta^2}{\omega\mu_0} - \frac{\partial^2}{\partial x^2} E_y\left(\frac{1}{\omega\mu_0}\right) = \omega\epsilon_0 n^2_{(x)} E_y$$

$$E_y\beta^2 - \frac{\partial^2}{\partial x^2} E_y = \omega^2\mu_0\epsilon_0 n^2_{(x)} E_y$$

$$\frac{d^2E_y}{dx^2} + \omega^2\mu_0\epsilon_0 n^2_{(x)} E_y - E_y\beta^2 = 0$$

$$\frac{d^2E_y}{dx^2} + \left[\omega^2\mu_0\epsilon_0 n^2_{(x)} - \beta^2\right] E_y = 0. \quad \text{--- (25)}$$

Let $k_0^2 = \omega^2\mu_0\epsilon_0$ $\quad k = \omega\sqrt{\mu_0\epsilon_0} = \quad k = \frac{\omega}{c}$ — (26)

$$\frac{d^2y}{dx^2} + \left[k_0^2 n^2_{(x)} - \beta^2\right] E_y = 0 \quad \text{--- (27)}$$

where $\sqrt{\mu_0\epsilon_0} = \frac{1}{c}$ is speed of
c is velocity of light in free space.

For the given refractive index profile $n = n^2(x)$ solution of eq. (27) [subject to the appropriate B.C, and continuity conditions] will give the field equations corresponding to TE. modes of wave guide.

Since. $E_y(x)$ is a tangential component, it should be continous at any discontinuity, further $\frac{dE_y}{dx}$ is $\propto H_z(x)$ (which is tangential component), it should be continous at any discontinuity. even

once $E_y(x)$ is known, $H_x(x)$ and $H_z(x)$ can be determined from eq(18) / eq. (20) eq.(22).

$n(x) = n_1$, for $x < \left(\frac{d}{2}\right)$ in film — (28).

$n(x) = n_2$ for $x > \left(\frac{d}{2}\right)$ in cover — 29

Substituting for $n(x)$ ie eq (28), and (29) in (27)

$$\frac{d^2E_y}{dx^2} + [k_0^2 n_1^2(x) - \beta^2] E_y = 0 \quad \text{— (30)} \quad |x| < d/2 \text{ in film.}$$

$$\frac{d^2E_y}{dx^2} + [k_0^2 n_2^2(x) - \beta^2] E_y = 0 \text{ —(31)} \quad |x| > d/2 \text{ in cover.} \quad \text{— (31)}$$

we will solve (30), (31) using appropriate boundary and continuity conditions.

ie E_x & H_z represent the tangential components on the planes $x = \pm(d/2)$, and they must be continous at $x = \pm d/2$ and since H_z is proportional to $\frac{dE_y}{dx}$, ie we must have

E_y and $\frac{dE_y}{dx}$ continous at $x = \pm d/2$.

For guidide wave, fields should decay in cover ie $x > d/2$ or we can say that fields mainly confined to films.

This is possible when

$$\beta^2 > k_0^2 n_2^2(x) \quad \text{— (32)}$$

If $\beta^2 < k_0^2 n_2^2(x)$ the solutions are oscillatory in region $|x| > (d/2)$.] then the fields corresponds to radiation modes of wave guide. These modes corresponds to rays that undergoes-refraction (rather than total internal reflection), at the film cover. So they leak away.

If $\beta^2 < k_0 n_1^2$ then. modes ~~are~~/ field exists inside the core. $|x| < d/2$.

To guide the modes the condition is

$$n_2^2(x) k_0 < \beta^2 < n_1^2(x) k_0 \quad \text{—— (33).}$$

using (33), eq. (31) & eq (32) can be rewritten as:

$$\frac{d^2E_y}{dx^2} + \alpha^2 E_y = 0 \quad — \quad |x| < \frac{d}{2} \text{ film} — (34).$$

$$\frac{d^2 E_y}{dx^2} - \gamma^2 E_y = 0 \quad — \quad |x| > \frac{d}{2} \text{ cover} — (35).$$

where $\alpha^2 = k_0^2 n_1^2 - \beta^2$ —— (36)

$\gamma^2 = \beta^2 - k_0^2 n_2^2$ —— (37)

for symmetric refractive index, system

$n^2(-x) = n^2(x)$.

for wave propagation α, γ should be positive.

i.e from (37) $\beta^2 > k_0^2 n_2^2$, & $\beta < k_0^2 n_1^2$ from (36).

In film $\frac{d^2 y}{dx^2} = -\alpha^2 E_y$. [if α is -ve, sol. decays, for γ sol. is not oscillatory [loss form]

$\frac{d^2 E_y}{dx^2} = \gamma^2 E_y$.

so the condition is (ie for total internal reflection).

$k_0 n_1^2 > \beta^2 > k_0^2 n_2^2$

$E_y(-x) = E_y(x)$ - symmetric modes.

$E_y(-x) = -E_y(x)$ - for antisymmetric modes.

For symmetric modes:

Sol of eq. (35) ~~are~~. and (34) are

$E_y = Ce^{\gamma x} + De^{-\gamma x}$ —— (38) - in cover.

$E_y = A\cos\alpha x + B\sin\alpha x$ - (39). - in film.

we will neglect the exponentially amplifying sol. in the region $|x| > \frac{d}{2}$

ie for $x > \frac{d}{2}$, $E_y = D e^{-\gamma x}$

$x < \frac{d}{2}$, $E_y = C e^{+\gamma s}$.

$E_y = A \cos \alpha x$ in film, $B = 0$.

$E_y = C e^{\gamma x}$ $\left[\begin{array}{l}\text{- for -ve } x\text{, values} \\ \text{- for +ve } x\text{, values}\end{array}\right]$ in cover.

$= D e^{-\gamma x}$

$+d/2$ —— $E_y = D e^{-\gamma x}$

Film $E_y = A \cos \alpha x$

$x = 0$ - - - - - - - -

$-d/2$ —— $E_y = C e^{+\gamma s}$.

Boundary conditions: E_y and H_z are continuous.

(1) $(E_y)_{film} = (E_y)_{cover}$ at $x = \frac{d}{2}$ (i) and $x = \left(-\frac{d}{2}\right)$ (ii)

(ii) $\left(\frac{dE_y}{dx}\right)_{film} = \left(\frac{dE_y}{dx}\right)_{cover}$ at $x = d/2$ (iii) and $x = (-d/2)$ (iv).

at $x = +(d/2)$

$(E_y)_{film} = A \cos\left(\alpha \frac{d}{2}\right)$ — [sub. for x].

$(E_y)_{cover} = D e^{-\gamma (d/2)}$

using B.C (i) we have.

$$A \cos\left(\alpha \frac{d}{2}\right) = D e^{-\frac{\gamma d}{2}} \quad \text{— (40)}$$

2nd B.Condition

$\left(\frac{dE_y}{dx}\right)_{film} = -A\alpha \sin \alpha x$.

$$\left(\frac{dE_y}{dx}\right)_{cover} = -D\gamma e^{-\left(\gamma \frac{d}{2}\right)} \quad \text{— (41)}$$

using B.C (ii) we have.

$$-A\alpha \sin\left(\frac{\alpha d}{2}\right) = -D\gamma e^{-\frac{\gamma d}{2}} \quad \text{—— (41)}$$

Dividing (41) by (40).

$$\tan\left(\frac{\alpha d}{2}\right) = \left(\frac{\gamma}{\alpha}\right) \quad - 42$$

at $x = \left(-d/2\right)$

$$(E_y)_{film} = A \cos(-\alpha x).$$

$$(E_y)_{cover} = C e^{+\gamma s}$$

using B.C (iii).

$$A \cos\left(\frac{\alpha d}{2}\right) = C e^{-\left(\frac{\gamma d}{2}\right)} \quad \text{—— (43)}$$

$$\left(\frac{dE_y}{dx}\right)_{film} = -A\alpha \sin \alpha x.$$

$$\left(\frac{dE}{dx}\right)_{cover} = +\gamma C e^{\gamma x}.$$

using BC(iv) we have.

$$-A\alpha \sin\left(\frac{\alpha d}{2}\right) = -C\gamma e^{-\gamma d/2} \quad \text{—— (44)}$$

Dividing (44) by (43)

$$\alpha \tan\left(\frac{\alpha d}{2}\right) = \gamma$$

$$\tan\left(\frac{\alpha d}{2}\right) = \left(\frac{\gamma}{\alpha}\right) \quad \text{—— (45).}$$

Consider

Add and subtract.

$$\frac{\gamma^2}{\alpha^2} = \left[\frac{\gamma^2}{\alpha^2} + 1 - 1\right]$$

$$=$$

$$\frac{\gamma^2}{\alpha^2} = \frac{\gamma^2+\alpha^2}{\alpha^2} - 1.$$

substitute for γ^2, α^2

$$\frac{\gamma^2}{\alpha^2} = \left[\frac{\cancel{\beta^2} - k_0^2 n_2^2 + k_0 n_1^2 - \cancel{\beta}}{\alpha^2} - 1\right]^{1/2} \quad — \quad 46.$$

$$\frac{\gamma^2}{\alpha^2} = \left[k_0^2 \left(\frac{n_1^2 - n_2^2 - 1)}{\alpha^2}\right]^{1/2}$$

multiply and divide both sides by $\left(\frac{d}{2}\right)^2$,

$$\frac{\gamma^2}{\alpha} = \frac{k_0^2 (n_1^2 - n_2^2) - \alpha^2}{\alpha}$$

$$\gamma^2\left(\frac{d}{2}\right)^2 = \left(\frac{d}{2}\right)^2 k_0^2 (n_1^2 - n_2^2) - \left(\frac{d}{2}\right)^2 \alpha^2.$$

$$\gamma\frac{d}{2} = \sqrt{\frac{V^2}{4} - \xi^2} \quad — \quad (47)$$

where $V = k_0 d (n_1^2 - n_2^2)^{1/2}$ — (48)

$\xi = \alpha\frac{d}{2} = (k_0^2 n_1^2 - \beta^2)^{1/2} d/2.$ — (49)

V is dimensionless wave guide parameter. equation (45) can be rewritten as

$$\xi \tan\xi = \left[\frac{V^2}{4} - \xi^2\right]^{1/2}. \quad — \quad (50).$$

variation of $\xi \tan\xi$ as a function of ξ is shown below. The point of intersection of the solid and dotted curves with the quadrant of a circle of radius $V_0 = V/2$ determine the propagation constants of the guide.

TE0, TE1, TE2.