CGS

Electric Polarizability

CGS units

1) $\chi_e = \frac{P}{E} = \frac{\epsilon - 1}{4\pi}$

ϵ → electric permittivity

2) $$\boxed{\frac{M}{\rho} \frac{\epsilon - 1}{\epsilon + 2} = \frac{4\pi}{3} N_A \alpha}$$

↓

Clausius-Mossotti eqn in CGS units

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and

$$\boxed{\frac{\epsilon_r - 1}{\epsilon_r + 2} = \sum_i \frac{N_i \alpha_i}{3\epsilon_0}}$$

↓

Clausius Mossotti eqn in SI unit

SI/MKS units

SI

1) $\chi_e = \frac{P}{\epsilon_0 E} = \frac{\epsilon}{\epsilon_0} - 1 = \epsilon_r - 1$

ϵ_0 → Permittivity of free space

2) $D = \epsilon_0 E + P = \epsilon_0 (1 + \chi_e) E$
$= \epsilon_0 (\epsilon_r) E$

~~3) $P = \chi_e \epsilon_0 E$~~

~~SI units~~

P → Polarization Density

3) Molecular polarizability

$$\boxed{p = \epsilon_0 \alpha E_{local}}$$

p → Induced dipole moment of individual molecule

$P = Np = N (\epsilon_0 \alpha E_{local})$

N ↓ No of molecules/unit volume

Here → α → Dimension of a volume (m^3)

4) But if we use $\boxed{p = \alpha E_{local}}$ SI

Here α has units $C \cdot m^2/V$

5) $\boxed{p = \alpha E_{local}}$ CGS

↓ In CGS α has dimension of volume but with a value divided by 4π

Clausius- Moscotti Eqn

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$D = \epsilon_0 \epsilon_r E = \epsilon_0 E + P$ (for isotropic / cubic medium)

$$\Rightarrow \epsilon_r = \frac{\epsilon_0 E + P}{\epsilon_0 E}$$

$$= 1 + \frac{P}{\epsilon_0 E}$$

$$= 1 + \chi_e$$

Also, $p = \alpha E_{loc}$

$P = Np = N\alpha E_{loc}$ = Total polarization

$$= \sum_i N_i \alpha_i E_{loc}(i)$$

$$= N\alpha \left(E + \frac{P}{3\epsilon_0} \right)$$

$$= N\alpha E + \frac{N\alpha P}{3\epsilon_0}$$

$$E_{loc} = E_0 + E_p = \cancel{E_p} + E_s = E + \frac{P}{3\epsilon_0}$$

lorentz eqn

$$P\left(1 - \frac{N\alpha}{3\epsilon_0}\right) = N\alpha E$$

$$\frac{P}{E} = \frac{N\alpha}{1 - \frac{N\alpha}{3\epsilon_0}}$$

$$\frac{P}{\epsilon_0 E} = \frac{N\alpha/\epsilon_0}{1 - \frac{N\alpha}{3\epsilon_0}}$$

mc

$$\epsilon_r - 1 = \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0}$$

$$\epsilon_r = 1 + \frac{N\alpha/\epsilon_0}{1 - \frac{N\alpha}{3\epsilon_0}} = \frac{1 - \frac{N\alpha}{3\epsilon_0} + \frac{N\alpha}{\epsilon_0}}{1 - \frac{N\alpha}{3\epsilon_0}}$$

$$= \frac{1 + \frac{2N\alpha}{3\epsilon_0}}{1 - \frac{N\alpha}{3\epsilon_0}}$$

$$\Downarrow$$

$$\boxed{\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{N\alpha}{3\epsilon_0}} \rightarrow \text{Clausius-Eqn Mosotti}$$

→ SI

In generalized form,

$$\boxed{\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{1}{3\epsilon_0}\sum_i N_i \alpha_i}$$

$n = \sqrt{\epsilon_0 \epsilon_r}$

↓

Refractive Index

At optical frequency ⟶ $\epsilon_r \rightarrow n^2$

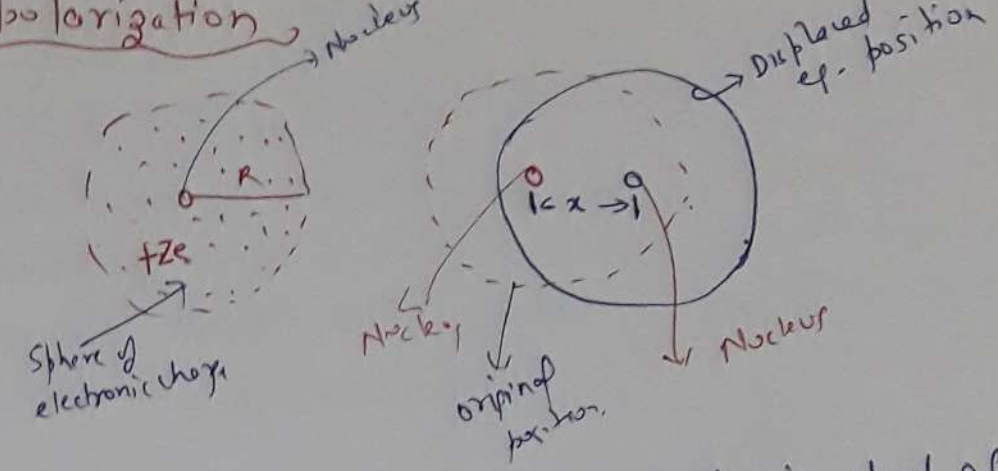
$n \rightarrow$ Refractive Index

$$\frac{n^2 - 1}{n^2 + 2} = \frac{1}{3\epsilon_0}\sum_i N_i \alpha_i \text{ (electronic)}$$

The polarization as well as dielectric constt. of a material at optical frequency result mainly from electronic polarizability.

Types of polarization

① **Electronic**



(Nucleus → charge +Ze) is sorrounded by electronic cloud of charge −Ze in a sphere of Radius R.

Charge density $= \rho = \dfrac{-Ze}{\frac{4}{3}\pi R^3} = \left(-\dfrac{3}{4}\right)\left(\dfrac{Ze}{\pi R^3}\right)$

E → External applied field.

Nucleus & e^- experience lorentz forces of magnitude $= qE = (Ze)E$ in opp. directions. Therefore they are pulled apart. & a Coloumb force develops b/w them which oppose the displacement & therefore net displacement $= 10^{-17}$ m for a field of 30 KV/m.

let, $x =$ Displacement

$$\text{lorentz field} = qE = (-Ze)E \qquad \text{—(1)}$$

$$\text{Coloumb Force} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} = (Ze)\,\frac{\text{Charge in sph. of radius } x}{4\pi\epsilon_0 x^2}$$

$$= \frac{(Ze)\left(\frac{4}{3}\pi x^3 . \rho\right)}{4\pi\epsilon_0 x^2} = \frac{Ze}{4\pi\epsilon_0 x^2}\left(\cancel{\tfrac{4}{3}}\cancel{\pi} x^3 . \cancel{-\tfrac{3}{4}}\frac{Ze}{\cancel{\pi}R^3}\right)$$

$$= \frac{-Ze}{4\pi\epsilon_0 x^2}\, x^3 . \frac{Ze}{R^3} = \frac{-Z^2e^2\, x}{4\pi\epsilon_0 R^3} \qquad \text{—(2)}$$

equating (1) & (2) as they are equal in equilibrium

$$(-\cancel{Ze})E = \frac{-(\cancel{Ze})(Ze)\,x}{4\pi\epsilon_0 R^3}$$

$$E = \frac{(Ze)x}{4\pi\epsilon_0 R^3} \Rightarrow \boxed{x = \frac{4\pi\epsilon_0 R^3 E}{Ze}} \quad -(3)$$

$$x \propto E$$

Displacement of e^- cloud is proportional to applied field E

Induced electric dipole moment $= p_e = qx = (Ze)x$

$$= (Ze)\left(\frac{4\pi\epsilon_0 R^3 E}{Ze}\right)$$

$$= 4\pi\epsilon_0 R^3 E$$

$$= \alpha_e E \qquad \text{where } \underset{\downarrow \text{ Electronic polarizability}}{\alpha_e = 4\pi\epsilon_0 R^3}$$

$$\boxed{p_e = \alpha_e E}$$

Dipole moment/vol. = Electronic polarization

$P_e = N p_e = N\alpha_e E$ where N = No of atoms/m^3

$P_e = \epsilon_0(\epsilon_r - 1)E$ — from below ↓

~~D = ε₀E +~~ P Three electric vectors – E, P, D

↓

~~ε_r ε₀~~E

① Electric field intensity $= \vec{E} \Rightarrow \lim_{q\to 0}\frac{F}{q}$ or $\vec{E} = \frac{q}{A\epsilon_0}$ (Gauss law)

② Polarization vector $= \vec{P} \Rightarrow$ Electric dipole moment induced per/vol.

Dipole moment of dielectric $= P \times \text{Vol} = P l A$ = charge × length $= \sigma_P A l$

on comparing $P = \sigma_P$

③ Electric displacement vector $= \vec{D} \Rightarrow \vec{E} = \vec{E_0} - \vec{E_p}$ (Resultant field in dielectric)

free charge $\quad E = \frac{q_0}{A\epsilon_0} - \frac{q'}{A\epsilon_0}$ (using Gauss law)

$q' \rightarrow$ surface charge density of induced charge

$$\epsilon_0 E = \frac{q_0}{A} - \frac{q'}{A} = \frac{q_0}{A} - P$$

$$\Rightarrow \frac{q_0}{A} = \epsilon_0 E + P$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

↳ Dielectric vector / flux density = No of lines of force rec'd / area

We can write $\Rightarrow$ $$\epsilon_0(\epsilon_r - 1)E = N\alpha_e E$$

$$\boxed{\epsilon_r - 1 = \frac{N\alpha_e}{\epsilon_0}}$$

ϵ_r → Dielectric constt. of material

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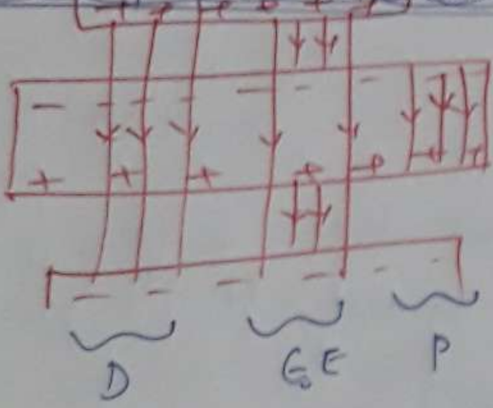
② Atomic / Ionic Polarization

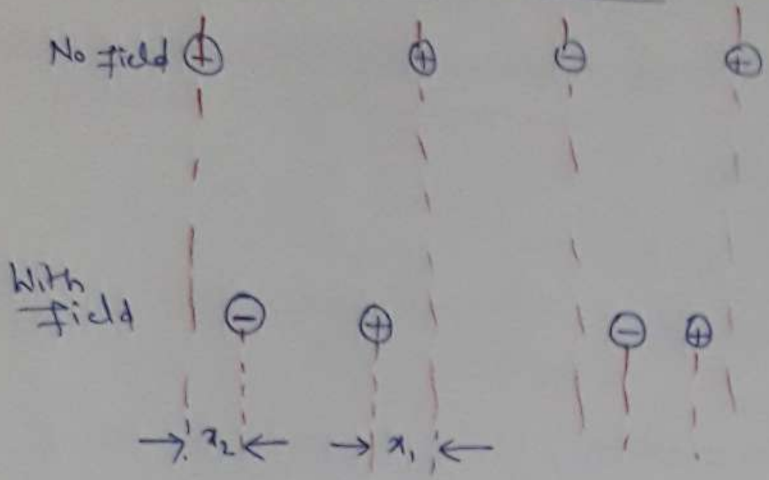


$\vec{E}$ → Connected with charges actually present (free or bound charges)

$\vec{D}$ → Connected with free charges only. Described by lines of displacement.
They begin & end on free charges.
Flux of $\vec{D}$ = free charge.

$\vec{P}$ → Connected with induced surface charge (polarization charge)
They begin & end on polarization charges / induced charges.

② Ionic / Atomic Polarization



Ex — NaCl, KBr, KCl, LiBr.

when an electric field is applied to ionic crystal, polarization arises due to displacement of +ve ions (away from field) & displacement of −ve ion (towards field) known as ionic/atomic pol.

p_i = Induced Dipole moment $= \alpha_i E$

$\alpha_i \longrightarrow$ Ionic polarizability.

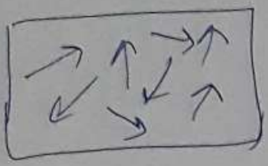
$N \longrightarrow$ No of dipoles / vol. due to polari.

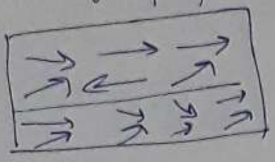
$P_i = N \alpha_i E$

✱ for most materials $\alpha_i < \alpha_e$ or $\alpha_i = 0.1\alpha_e$

③ Dipolar / Orientational Polarization

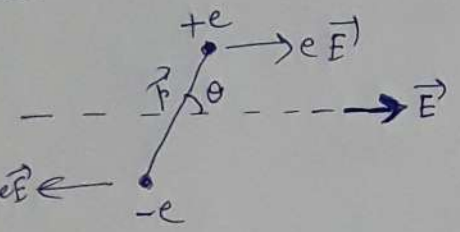
It is produced only in case of polar molecules like H_2O, HCl etc.
On applying ext. electric field → Dipoles experience a torque and try to allign ∥ to field, which results in rotation of dipoles


$E = 0$
(Random dipoles)


$E \neq 0$
(Dipoles rotating)

Allignment of dipole is opposed by the thermal agitation which tends to randomize the dipoles. Due to this, all calculations are done under thermal equillibrium.

Ⓐ Consider a dipolar system → placed in uniform electric field



Torque acts on each dipole (tries to orient the dipole along $\vec{E}$)
↓
But ordering effect + Thermal motion of dipoles
↓
New equilli state is reached.
↓
Produce net polarization which we are going to calculate.

Ⓑ P.E. of a dipole $= -pE\cos\theta = -\vec{p}\cdot\vec{E}$

Ⓒ Prob for dipole to make an angle b/w θ & θ + dθ with field $\propto 2\pi\sin\theta\, d\theta\, \exp\left(\frac{pE\cos\theta}{KT}\right)$ → Stat. Mechan.

$\propto$ probability

Ⓓ $2\pi\sin\theta\, d\theta$ → Solid angle b/w θ & θ + dθ

ⓔ No of dipoles with orientation b/w θ & θ + dθ ∝ Prob.

ⓕ Dipole of moment p makes contribution $= p\cos\theta$ (cos component)

(g) Total contribution to p = $(p\cos\theta) \times$ Prob.

$$p_{Total} = (p\cos\theta)\,(2\pi \sin\theta \, d\theta \, e^{pE\cos\theta/KT})$$

(h) Avg. contribution ~~[illegible]~~/dipole $= \bar{p} = \dfrac{\int_0^{\pi} p_{Total}}{\int_0^{\pi} \text{Prob}} = \dfrac{\int_0^{\pi} p\cos\theta \, 2\pi \sin\theta \, d\theta \, e^{pE\cos\theta/KT}}{\int_0^{\pi} 2\pi \sin\theta \, e^{pE\cos\theta/KT}}$

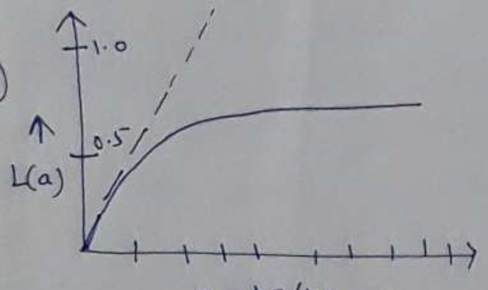
$\theta = 0 \rightarrow \parallel$ Allignment
$\theta = \pi \rightarrow$ Antill allignment

(i) put $a = pE/KT$, $x = a\cos\theta$, $dx = -a\sin\theta \, d\theta$

$$\bar{p} = \left(p \int_{-a}^{a} x e^{x} dx \Big/ a \int_{-a}^{a} e^{+x} dx \right) = p\left(\frac{e^{a}+e^{-a}}{e^{a}-e^{-a}} - 1 \right)$$

$$= p\left(\coth a - \frac{1}{a}\right) = p\,L(a)$$ $\longrightarrow$ langevin functio

(J)



Case-I : a is very large ⇒ At very high field strength | Low temp
$L(a) \rightarrow$ Saturation Value or Unity.
⇒ Max. allignment of dipoles along the direction of field.

Case-II : $a \ll 1$ or $pE \ll KT$ ⇒ Fields not too large & temp not too low

$L(a) \Rightarrow a/3$

$$\bar{p} = p \cdot \frac{a}{3} = p\left(\frac{pE}{KT}\right)\frac{1}{3} = \frac{p^2 E}{3KT}$$

α_D = dipolar polarizability $= \bar{p}/E = p^2/3KT$

⇒ Dipole/ orientational polarization is temp dependent.

(k) <u>Langevin- Debye equation</u> → $\alpha = \alpha_{elect} + \alpha_{dip}$ ~~[illegible]~~ $+ \alpha_{ionic}$

$\alpha = \alpha_0 + \alpha_{dip.} = \alpha_0 + \frac{p^2}{3kT}$ → langevin-debye eqn.

= total polarizability

Due to this Clausius -Mossotti eqn → $\frac{M}{\rho}\frac{\epsilon-1}{\epsilon+2} = \frac{4\pi}{3}N_A\left(\alpha_{ei} + \frac{p^2}{3kT}\right)$

~~Molar polarizability~~ ~~$\frac{M}{\rho}\frac{\epsilon-1}{\epsilon+2}$ vs $1/T$~~

Behaviour of Dielectrics in A.C. field

(1) Dielectric constant becomes complex. Imaginary part of the complex dielectric constt. determines the dielectric losses of material.

(2) When dielectric is placed in Alternating field, the orientation of dipoles & hence polarization will tend to reverse, when the polarity of field changes

- If frequency is low ($< 10^6$ c/s) there is no significant lag in polarization with alterations of field. The permittivity is independent of frequency & has magnitude as in static field
- If freq is ↑ dipoles will not be able to rotate rapidly & oscillation will lag behind field.
- with further ↑ of freq, permanent dipoles in medium will be unable to follow the field & contribution to static permittivity from this molecular process (i.e.) orientation polarization stops (dipolar freq range → Radio → $10^6 - 10^{11}$ Hz of em spectrum
- At still higher freq (infra-red range, $10^{11} - 10^{14}$ Hz) the relatively heavy positive & -ve ions can't follow field alterations & contribution to permittivity from Atomic or ionic polarization stops & only electronic pol. remains.

(3) Thus permittivity of dielectric material decreases with increase in frequency & this phenom. is called anamalous dielectric dispersion.

(A) Dielectric Absorption → Dispersion arising during the transition from ~~at~~ full atomic polarization at radio freq to negligible atomic polarization at optical freq is called dielectric absorption.

(B) Dielectric relaxation → Dispersion arising during transition from full orientational polarization at zero/low freq to negligible orientational polarization at high freq is called dielectric relaxation

Polarization → It is greatly effected by freq. of applied voltage. Electronic polarization occurs at all frequencies ranging upto optical frequencies ($\sim 10^{15}$ Hz).

Ionic polarization occur at upto infrared range (10^{13} Hz)

orientational polariz. occur when freq in audio range (10^{11} Hz)

Space charge polari. at freq upto 100 Hz.

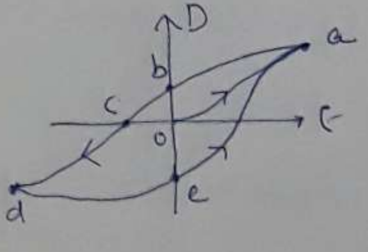
Space charge Polarization → It occurs in multiphase dielectric materials such materials, in high $\vec{E}$ field (at high temp), ~~behaves as a~~

↓ Charges get accumulated at interface or electrodes because of sudden change in conductivity.

(freq dependence of dielectric constt.

Dielectric losses

when a dielectric is placed in an alternating field there is an electrical energy loss in dielectric which is converted into heat. This is called "loss of power".



phenomenon of displacement vector $\vec{D}$ lagging behind $\vec{E}$ is called Hysteresis.

Curve b/w $\vec{D}, \vec{E}$ ⇒ Hyst. curve

Area under $\vec{D}, \vec{E}$ curve ⇒ loss of energy

⇒ Dielectric loss

Derivation: let $\vec{E} = \vec{E}_0 \cos\omega t$

$$\vec{D} = \vec{D}_0 \cos(\omega t - \delta) = \bar{D}_0 \cos\delta \cos\omega t + \bar{D}_0 \sin\delta \sin\omega t$$

$$= \bar{D}_1 \cos\omega t + \bar{D}_2 \sin\omega t \quad \text{--}(1)$$

, where $\vec{D}_1 = \vec{D}_0 \cos\delta$, $\vec{D}_2 = \vec{D}_0 \sin\delta$, δ = phase Angle.

* for most dielectrics $\vec{D}_0 \propto \vec{E}_0$
* But ratio $(\vec{D}_0/\vec{E}_0)$ is freq dependent.

let $\epsilon_r' = \left|\frac{\bar{D}_1}{E_0}\right| = \left|\frac{\bar{D}_0}{E_0}\right| \cos\delta$ → freq. dependent dielectric constt.

$\epsilon_r'' = \left|\frac{\bar{D}_2}{E_0}\right| = \left|\frac{\bar{D}_0}{E_0}\right| \sin\delta$ →

let $\epsilon_r^* = \epsilon_r' + i\epsilon_r''$

$\vec{D} = (\epsilon\vec{E}) = (\epsilon_r^*) \times (E_0 e^{i\omega t}) = \epsilon_r^* \vec{E}_0(\cos\omega t + i\sin\omega t)$ (2)

let $\tan\delta = \frac{\epsilon_r''}{\epsilon_r'}$ (As ϵ_r' ϵ_r'' are freq. dependent so phase angle δ is also freq. dependent)

Now current density in capacitor $= \frac{d\vec{D}}{dt} = \vec{J}$

$$\vec{J} = \frac{d}{dt}(\bar{D}_1\cos\omega t + \bar{D}_2\sin\omega t)$$

$$= (-\vec{D}_1\sin\omega t + \vec{D}_2\cos\omega t)\omega$$

Energy dissipated / sec in dielectric $= W = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\vec{J}\cdot\vec{E}\,dt$

$$= \frac{1}{T_1}\int_0 \vec{J}\cdot\vec{E}\,dt$$

$\Downarrow$

$$W = \frac{\omega}{2\pi}\left[\int_0^{2\pi/\omega}\omega(-\bar{D}_1\sin\omega t + \bar{D}_2\cos\omega t)\cdot E_0\cos\omega t\,dt\right]$$

$$= \frac{\omega}{2\pi}\omega\int_0^{2\pi/\omega}(-D_1\sin\omega t\underset{0}{\cancel{\cos\omega t}}E_0\,dt + D_2E_0\cos^2\omega t\,dt)$$

$$= \frac{\omega^2}{2\pi}\times\bar{D}_2\bar{E}_0\times\int_0^{2\pi/\omega}\cos^2\omega t\,dt = \frac{\omega^2}{2\pi}\times\bar{D}_2\bar{E}_0\times\left(\frac{\pi}{\omega}\right)$$

$$= \frac{\omega}{2}\bar{D}_2\bar{E}_0$$

$$= \frac{\omega}{2}(\vec{D}_0\sin\delta)\vec{E}_0 = \frac{\omega}{2}\bar{D}_0\bar{E}_0\sin\delta = \frac{\omega}{2}\bar{E}_0(\bar{D}_0\sin\delta)$$

$$W = \frac{\omega}{2}\bar{E}_0(\bar{E}_0\epsilon_r'') = \frac{\omega}{2}E_0^2\epsilon_r''$$

$\Rightarrow W \propto \varepsilon_r''$ or $\sin\delta$

$\sin\delta \rightarrow$ Loss factor | $\tan\delta \rightarrow$ Loss factor (for small values of δ)

$\delta \rightarrow$ losss Angle

Note: Dielectric loss at low frequencies is mainly due to d.c. resistivity but at high frequencies dielectric loss is mostly due to dipole rotations.

Lorentz- Lorentz formula / Relation b/w Dielectric Constt & refractive index.

$V = \frac{1}{\sqrt{\mu\epsilon}}$ = Velocity of propagation of em in an unbounded medium.

$\mu \rightarrow$ Mag. permeability of medium

$\epsilon \rightarrow$ Absolute permittivity

$\mu_0 \rightarrow$ Permeability for non-magnetic media. $\Rightarrow$ velocity $= \frac{1}{\sqrt{\mu_0\epsilon}}$

η = Refractive index of medium $= \frac{\text{velocity of waves in vaccum}}{\text{'' \; '' \; '' \; '' medium}}$

$$= \frac{1/\sqrt{\mu_0\epsilon_0}}{1/\sqrt{\mu_0\epsilon}} = \sqrt{\epsilon/\epsilon_0} = \sqrt{\epsilon_r} \quad \text{—— (1)}$$

We also know that

$$\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{N\alpha}{3\epsilon_0} \quad \text{(SI units)} \quad \text{—— (2)}$$

From (1) & (2)

$$\boxed{\frac{\eta^2 - 1}{\eta^2 + 2} = \frac{N\alpha}{3\epsilon_0}} \rightarrow \text{Lorentz- lorentz formula}$$

Classical Theory of electronic Polarizability

An electron bound harmonically to an atom will show resonance absorption at a freq $\omega_0 = \sqrt{\beta/m}$ where, β = force const.

m = Mass of e^-

x = Displacement of e^- under $\vec{E}_{loc}$

Steps
⇒ $-e\bar{E}_{loc} = \beta x = m\omega_0^2 x$ = force. (static field)

Static Electronic polarizability = $\alpha_e = \frac{p}{E_{loc}} = \frac{(-ex)}{\left(-\frac{m\omega_0^2 x}{e}\right)}$

$$\boxed{\alpha_e = \frac{e^2}{m\omega_0^2}} \rightarrow (1)$$

Time-dependent

$\vec{E}_{loc} \longrightarrow \vec{E}_{loc}\sin\omega t \Rightarrow x \to x_0 \sin\omega t$

We assume system to be SH oscillator. Eqn of motion in $\vec{E}_{loc}\sin\omega t$ is →

$$m\frac{d^2x}{dt^2} + m\omega_0^2 x = -e\bar{E}_{loc}\sin\omega t$$

$$m\frac{d^2(x_0\sin\omega t)}{dt^2} + m\omega_0^2(x_0\sin\omega t) = -e\bar{E}_{loc}\sin\omega t$$

$$m x_0(-\omega^2)\sin\omega t + m\omega_0^2 x_0 \sin\omega t = -e\bar{E}_{loc}\sin\omega t$$

$$m x_0(-\omega^2 + \omega_0^2) = -e\bar{E}_{loc}$$

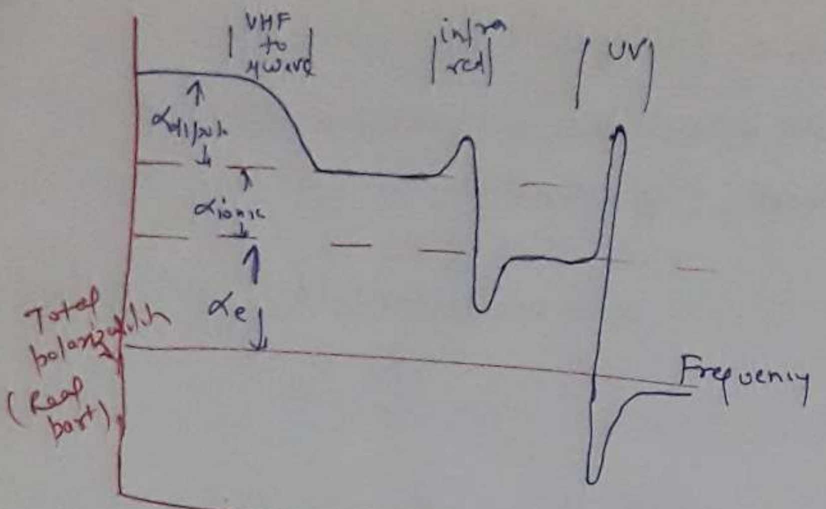
$$\boxed{x_0 = \frac{-eE_{loc}}{m(-\omega^2+\omega_0^2)}} \quad — (2)$$

~~Using (1) & (2)~~

For $\underline{x_0}$ → Dipole moment amplitude $= p_0 = -ex_0 = \frac{e^2 E_{loc}}{m(\omega_0^2-\omega^2)}$

α_e = Electronic pola. $= \frac{p_0}{E_{loc}} = \frac{e^2/m}{(\omega_0^2-\omega^2)}$

↓
Depends on ω freq



Frequency dependence of various polarizabil.

If $Z \to$ Electrons/atom

$N \to$ Atoms/unit volume.

χ_e = Electric susceptibility $= \dfrac{NZe^2/\epsilon_0 m}{(\omega_0^2-\omega^2)}$

$\eta^2 = 1+\chi_e = 1 + \dfrac{NZe^2/\epsilon_0 m}{\omega_0^2-\omega^2} \longrightarrow$ Dispersion relation curve

* when frequency of applied field is much greater than inverse of relaxation time for a particular polarization process, then that polarization process fails & so it does not contribute to polarizability.

Complex Dielectric Constt.

A complex dielectric constt is used to describe the ~~situation~~ diel. constt. of periodic variation variation of $\vec{E}$ field. It is written as —

$$\boxed{\epsilon = \underbrace{\epsilon'}_{\text{Real}} + i\underbrace{\epsilon''}_{\text{Imag.}}}$$

Real → Stored energy

Imag. → Dielectric loss factor → (Also gives rise to phase diff b/w $\vec{E}$ & resulting polarizatn)

(I) We consider Non-Polar Solid → No permanent dipole
→ Polarization is only electronic / ionic

(II) It is subjected to time varying field in which

$$\vec{E} = \text{static field} \xrightarrow[\text{to}]{\text{changes}} \vec{E}(t) = E_0 e^{i\omega t}$$

$$\left.\begin{array}{l} \vec{E}_{loc} = \vec{E} + \dfrac{4\pi P}{3} \; (CGS) \\ \text{or} \\ \vec{E}_{loc} = \vec{E} + \dfrac{\vec{P}}{3\epsilon_0} \; (SI) \end{array}\right\} \xrightarrow[\text{to}]{\text{changes}} \begin{array}{l} \vec{E}_{loc}(t) = E(t) + \dfrac{4\pi}{3} P(t) \\ \vec{E}_{loc}(t) = \vec{E}(t) + \dfrac{P(t)}{3\epsilon_0} \end{array}$$

$$\vec{p} = \alpha \vec{E} \xrightarrow[\text{to}]{\text{changes}} P(t) = (\alpha_e + \alpha_i) E_{loc}(t)$$

$$\alpha_i \; (\text{static}) \longrightarrow \alpha_i^* = \alpha_i' - i\alpha_i''$$

$$\alpha_e \; (\text{static}) \longrightarrow \alpha_e^* = \alpha_e' - i\alpha_e''$$

$$\vec{E}_{loc} \longrightarrow E_{loc}(t) = E_{0loc}^* e^{i\omega t}$$

$$\epsilon_r \longrightarrow \epsilon_r^*$$

$$P = \epsilon_0(\epsilon_r - 1)E \longrightarrow P(t) = \epsilon_0 \left[\text{Re}\left\{(\epsilon_r^* - 1) E(t)\right\}\right] \quad \text{eq} \; ①$$

$* \rightarrow$ Complex Quantity

(III)

$$\vec{E}_{loc}(t) = E(t) + \frac{P(t)}{3\epsilon_0} = \text{Re}(E_0 e^{i\omega t}) + \frac{1}{3\epsilon_0} \times \epsilon_0 \, \text{Re}\left((\epsilon_r^* - 1)E_0 e^{i\omega t}\right)$$

$$= \text{Re}\left[E_0 e^{i\omega t}\right] + \frac{1}{3} \text{Re}\left[(\epsilon_r^* - 1) E_0 e^{i\omega t}\right]$$

$$= \mathrm{Re}\left[\left(1 + \frac{\epsilon_r^* - 1}{3}\right) E_0 e^{i\omega t}\right]$$

$$= \mathrm{Re}\left[\left(\frac{\epsilon_r^* + 2}{3}\right) E_0\, e^{i\omega t}\right]$$

$$= \mathrm{Re}\left[E_{0\,loc}^*\, e^{i\omega t}\right] \quad \text{where} \quad E_{0\,loc}^* = \left(\frac{\epsilon_r^* + 2}{3}\right) E_0$$

(IV) Again using $p = \alpha E$ →

→ ~~$P(t) = N\,\mathrm{Re}\left((\alpha_e^* + \alpha_i^*) E_{0\,loc}^*\, e^{i\omega t}\right)$~~

we get -

$$P(t) = N\,\mathrm{Re}\left[(\alpha_e^* + \alpha_i^*) \cdot \frac{E_0}{3}(\epsilon_r^* + 2) \cdot e^{i\omega t}\right] \quad \text{— eq ②}$$

Comparing eq ① & eq ② and rearranging -

$$\frac{\epsilon_r^* - 1}{\epsilon_r^* + 2} = \frac{N}{3\epsilon_0}(\alpha_e^* + \alpha_i^*)$$

→ This eqn relates ϵ_r^* under A.C. conditions to atomic quantities of the material. It is also applicable for lorentz field conditions.

→ ϵ_r^* → It depends on α_e^*, α_i^*

→ $\epsilon_r^* = \epsilon_r' - i\epsilon_r''$

→ Nature of variation of ϵ_r', ϵ_r'' with freq. depends on freq range ~~on freq~~ under consideration.

→ ϵ_r^* remains real upto μwave.

→ ~~Several solids dielectrics contains ions~~

on Comparin(1) & (2))

$$N\,\mathrm{Re}\left[(\alpha_e^* + \alpha_i^*)\frac{E_0}{3}(\epsilon_r^* + 2)e^{i\omega t}\right] = \epsilon_0\,\mathrm{Re}\left[(\epsilon_r^* - 1)\,E(t)\right]$$

$$\frac{N}{3\epsilon_0}(\alpha_e^* + \alpha_i^*) = \frac{\epsilon_r^* - 1}{\epsilon_r^* + 2}$$

Normal & Anomalous Dispersion

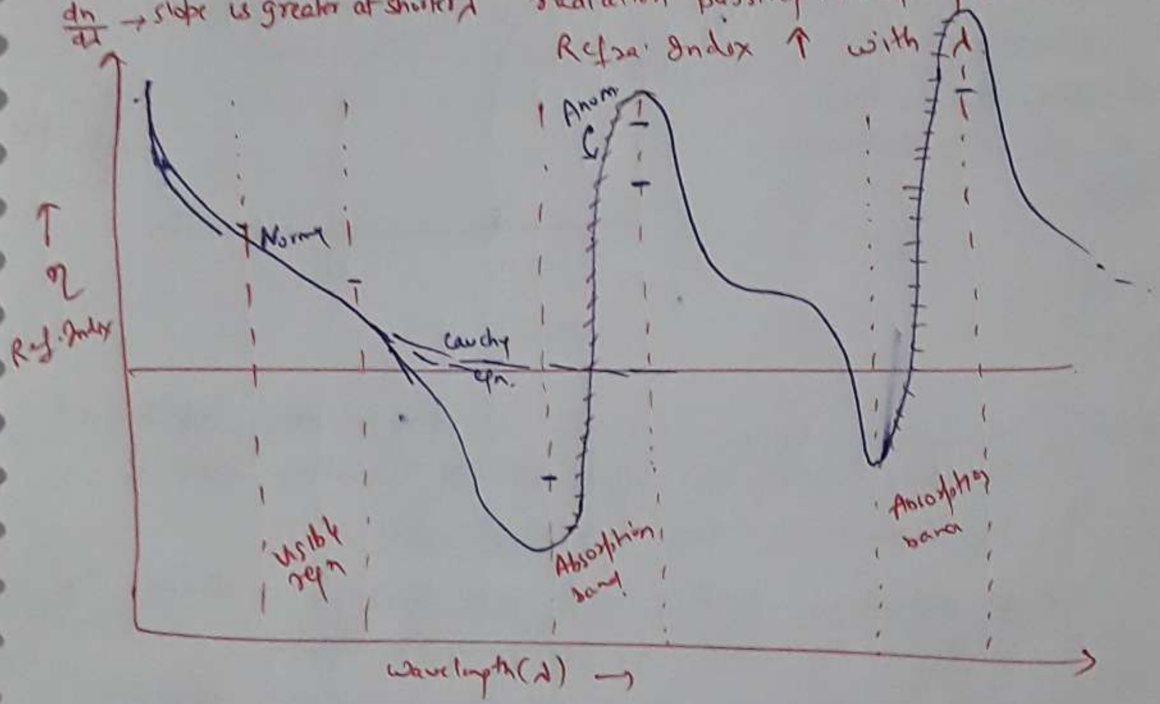
① When refractive index of a medium varies with λ (i.e. frequency), it is called dispersion and medium is called Dispersive medium.

Normal dispersion

Anomalous dispersion.

↓
Refrac' Index decreases with λ
$\frac{dn}{d\lambda}$ → slope is greater at shorter λ

Over small λ ranges, there is often ↑ of refractive index due to an ↑ absorption of radiation passing through medium.
Refra' Index ↑ with λ



Cauchy's Eqn: Normal dispersion is explained by Cauchy's empirical eqn

$$\eta^2 = A + \frac{B}{\lambda} + \frac{C}{\lambda^4}$$

A, B, C → Constt. depending on medium & decreasing rapidly in magnitude as we go to higher order terms.

→ Constt. are positive → η decreases as λ increases.

η → Max. at violet
→ least at red of visible region of spectrum

$$\frac{dn}{d\lambda} = \frac{-2B}{\lambda^3} \cdot \frac{1}{2n} = -\frac{B}{\lambda}\left(A + \frac{B}{\lambda^2}\right)^{-1/2} \cong \frac{-AB}{\lambda^3}$$

(upto A, B only)

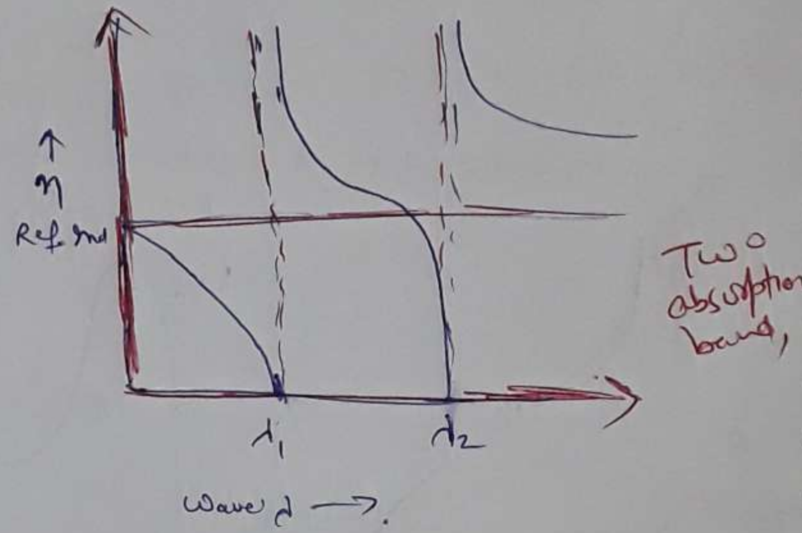
Dispersion $\propto \frac{1}{\lambda^3}$

Slope = -ve.

Magnitude of slope = Decreases ↓ as λ ⇑

Sellmeier's Formula :

1871, sellemeier gave dispersion formula on the basis of elastic solid theory of light



Assumption : Medium contains elastically bound particles capable of vibrating with natural freq of vibration say ν_0 in the absence of any periodic force

- If radiation of this freq. pass through medium, the particle resonate & energy is absorbed.
- for radiation of other frequencies particle execute forced oscillations, the amplitudes increasing as freq. of radiation approaches resonant freq.
- these vibration cause a change in velocity of radiation if all the elastically bound particles in medium have a natural freq. corresponding to radiation of λ_0 in vacum, then sellmeir's give $n^2 = 1 + \frac{A\lambda^2}{\lambda^2 - \lambda_0^2}$

This eqn is only for one absorption band. In case of several bands $\Rightarrow$ $n^2 = 1 + \sum_K \frac{A_K \lambda^2}{\lambda^2 - \lambda_K^2}$

A_K → ~~No of p~~ It is proportional to no of particle/Volume whose natural freq corresponds to λ_K

→ There exists particles with several diff natural frequencies

→ Sellmeier formula fails in the region of absorption band, bcoz η is ∞ & discontinuous at λ_1, λ_2 etc.

→ In regions far away from absorption bands it gives better result

→ If there is one absorp. band, at considerable distance to a short λ side of region where dispersion is reqd ($\lambda_0 \lll \lambda$) then —

$$n^2 = 1 + A\left(1 - \frac{\lambda_0^2}{\lambda^2}\right)^{-1}$$

$$= 1 + A + \frac{A\lambda_0^2}{\lambda^2} + A\frac{\lambda_0^4}{\lambda^4} \longrightarrow \text{Same as Cauchy's formula}$$

Theory of Normal & ~~(Con~~ Anomalous Dispersion (em Theory)

Theory of dispersion based on em theory → By lorentz.

Assumption:

(1) when em wave passes through a medium, atomic (molecular) dipoles are created (due to relative displa. of e^- & nuclei). Dipole moment is induced in gas molecule.

(2) Dipoles execute forced vibrations (due oscillatory $\vec{E}$ field)

(3) Interactions b/w dipoles are negligible. (free Dipoles as for gas)

(4) e^- bound to nucleus by restoring force.

(5) Over an atom/molecule $\vec{E}$ is constt. in space —

$$\vec{E} = \vec{E_0}\exp(-i(\omega t - \vec{k}\vec{r})) = E_0 e^{-i\omega t}$$

(6) force $= q\vec{E}$

No of forces acting on e^- → 3

1st → $q\vec{E}$ (electric force)

2nd → elastic restoring force ∝ Displacement $\vec{r}$ from equilib.
$= -k\vec{r}$ (k is +ve)

3rd → Damping frictional force against motion of e^- ∝ velocity
$= -\alpha \frac{d\vec{r}}{dt}$ (α is +ve const. of proportion.)

Newton's law ⇒

$$m\frac{d^2\vec{r}}{dt^2} = q\vec{E} - \alpha\frac{d\vec{r}}{dt} - k\vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} + \gamma\frac{d\vec{r}}{dt} + \omega_0^2\vec{r} = \frac{q}{m}\vec{E}$$

(where $\gamma = \alpha/m$)
($\omega_0^2 = k/m$)

Solution ⇒

$$\boxed{\vec{r} = \frac{\left(\frac{q}{m}\right)\vec{E}_0 e^{i\omega t}}{(\omega_0^2 - \omega^2 - i\gamma\omega)}} \quad \text{—(1)}$$

$$\vec{P} = \text{Polarisation} = Nq\vec{r} = \frac{Nq \times \frac{q}{m}\vec{E}_0 e^{i\omega t}}{(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

$$\boxed{\vec{P} = \frac{Nq^2\vec{E}}{m}\frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)}} \quad \text{—(2)}$$

↓
ω_0 = Natural freq. of oscillation for all oscillators

If N molecules/vol → In each N molecule → f_1 oscillator having ω_1

→ f_2 have ω_2

→ f_3 have ω_3 freq

⇒ Medium absorbs more than one spectral line

$$\cancel{\alpha = \frac{P}{E} = \frac{Nq^2}{m}\sum_k} \Rightarrow \vec{P} = \frac{Nq^2\vec{E}}{m}\sum_K\frac{f_K}{(\omega_K^2 - \omega^2 - i\gamma_K\omega)}$$

from electrical suscepti ⇒ $P = \epsilon_0 \chi_e E$

$$\frac{P}{\epsilon_0 E} = \chi_e$$

$$\epsilon_r = 1+\chi_e = 1 + \frac{P}{E}\frac{1}{\epsilon_0}$$

$$\Rightarrow \epsilon_r = 1 + \frac{1}{\epsilon_0}\frac{Nz^2}{m}\sum_k \frac{f_k}{(\omega_k^2 - \omega^2 - i\gamma_k\omega)}$$

$$\boxed{(n^*)^2 = \epsilon_r = 1 + \frac{1}{4\pi\epsilon_0}\frac{4\pi Nz^2}{m}\sum_k \frac{f_k}{(\omega_k^2 - \omega^2 - i\gamma_k\omega)}} \quad \text{eq ③}$$

eq③ → <u>Dispersion formula</u>

⇒ Ref. index is complex → Absorption of energy is in medium

→ Ref Index is freq dependent.

NORMAL DISPERSION

It is region away from natural frequencies (or Absorption frequencies)

- $\gamma_k\omega$ term in eq③ → 0
- eq③ reduces to $n^2 = 1 + \frac{1}{4\pi\epsilon_0}\frac{4\pi Nz^2}{m}\sum_k \frac{f_k}{(\omega_k^2 - \omega^2)}$

⇒ n is real & ↑ with freq. of incident wave

It is normal dispersion

→ $\omega = 2\pi\nu = 2\pi\frac{c}{\lambda}$ $\Rightarrow n^2 = 1 + \frac{1}{4\pi\epsilon_0}\frac{4\pi Nz^2}{m}\sum_k \frac{f_k}{\left(\frac{4\pi^2c^2}{\lambda_k^2} - \frac{4\pi^2c^2}{\lambda^2}\right)}$

$$n^2 = 1 + \frac{1}{4\pi\epsilon_0}\frac{4\pi Nz^2}{m}\sum_k \frac{f_k}{4\pi^2c^2}\frac{\lambda^2\lambda_k^2}{(\lambda^2 - \lambda_k^2)}$$

Sellmeier's eqn ← $\boxed{n^2 = 1 + \sum_k \frac{A_k\lambda^2}{(\lambda^2 - \lambda_k^2)}}$ $A_k = \frac{Nz^2\lambda_k^2 f_k}{\epsilon_0 m \times 4\pi^2c^2}$

21 sep

$\lambda >> \lambda_K$

$$n^2 = 1 + \sum_K A_K \left(1 - \frac{\lambda_K^2}{\lambda^2}\right)^{-1}$$

$$= 1 + \sum_K A_K + \sum_K A_K \frac{\lambda_K^2}{\lambda^2} + \sum_K A_K \frac{\lambda_K^4}{\lambda^4} + \cdots$$

$$n^2 = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \cdots \quad \rightarrow \text{eq} ④$$

where, $A = 1 + \sum_K A_K$ = coefficient of refraction

$B = \sum_K A_K \lambda_K^2$ = coefficient of dispersion

$C = \sum_K A_K \lambda_K^4$

eq ④ → Cauchy's Dispersion formula.

Differentiate eq ④ ➜ $2n \frac{dn}{d\lambda} = -\frac{2B}{\lambda^3} - \frac{4C}{\lambda^5}$

$$= -\frac{2}{\lambda^3}\left(B + \frac{2C}{\lambda^2} + \cdots\right)$$

$$\frac{dn}{d\lambda} = -\frac{1}{\lambda^3}\left(B + \frac{2C}{\lambda^2} + \cdots\right)\frac{1}{n}$$

$$\frac{dn}{d\lambda} = -\frac{1}{\lambda^3}\left(B + \frac{2C}{\lambda^2} + \cdots\right)\left(A + \frac{B}{\lambda^2} + \cdots\right)$$

$$\boxed{\frac{dn}{d\lambda} = -\frac{BA}{\lambda^3} = -\frac{k}{\lambda^3}} \rightarrow \text{Dispersive Power}$$

Dispersive Power $\propto \frac{1}{\lambda^3}$

–ve sign ⇒ Slope of dispersive curve is –ve

Anomalous Dispersion (It is the region of natural freq.)

As long as impressed frequencies do not include a natural freq. of e^-, the dispersion is always normal.

We assume (for simplicity) ⟶ There is one natural freq.

i.e. $\omega_K = \omega_0$

$$n^{*2} = 1 + \frac{1}{4\pi\epsilon_0}\frac{4\pi N q^2}{m}\frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

$$n^* = \left[1 + \frac{1}{4\pi\epsilon_0}\frac{4\pi N q^2}{m}\cdot\frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)}\right]^{1/2}$$

$$n^* = 1 + \frac{1}{4\pi\epsilon_0}\frac{2\pi N q^2}{m}\frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

$\downarrow$ Rationalize

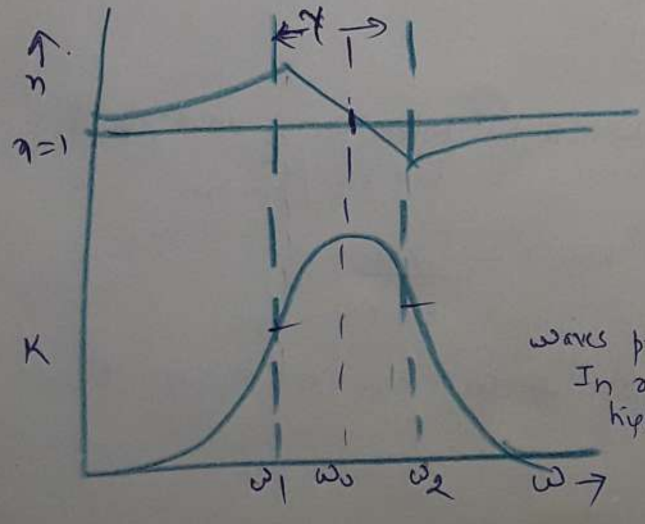
$$n^* = 1 + \frac{2\pi N q^2}{4\pi\epsilon_0 m}\left\{\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}\right\} + i\,\frac{2\pi N q^2}{4\pi\epsilon_0 m}\left\{\frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\right\}$$

$$= n + iK$$

where,

$$n = 1 + \frac{2\pi N q^2}{4\pi\epsilon_0 m}\left\{\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}\right\}$$

$$K = \frac{2\pi N q^2}{4\pi\epsilon_0 m}\left\{\frac{\gamma\omega}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}\right\}$$



$K \rightarrow$ max at ω_0

width at half max $= \gamma$

Imaginary part correspond to absorption of em waves propagating through gas. ~~Thus~~

In region where n changes rapidly gas is relatively highly absorbing

$$n^* = 1 + \frac{1}{4\pi\epsilon_0}\frac{2\pi N q^2}{m}\sum_k \frac{f_k(\omega_k^2 - \omega^2 + i\gamma\omega)}{(\omega_k^2 - \omega^2) + \gamma^2\omega^2}$$

$\downarrow$ many resonant freq.

Plot of real (n) & imag. (K) part of n^* as function of ω

Optical Phenomenon

Solids → Transparent
→ opaque
→ Highly reflecting
→ Bend em waves incident on them

} → These properties are studied by $\epsilon_r(\omega)$ of solid

$$\vec{E} = \vec{E}_0 e^{-i\omega\left(t - n^*(\omega)\frac{z}{c}\right)}$$

→ $\vec{E}$ field for em waves travelling along Z-direction

$n^*(\omega)$ = complex ref. index
$= n + ik = \sqrt{\epsilon_r(\omega)}$ → (1)

But $n^*(\omega) = c/v$

But

$$\epsilon_r^*(\omega) = \epsilon_r'(\omega) + i\,\epsilon_r''(\omega) \quad \text{— (2)}$$

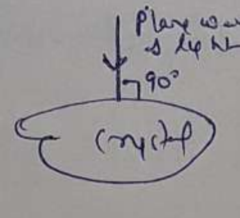
Using (1) & (2)

$$\epsilon_r'(\omega) = n^2 - K^2$$
$$\epsilon_r''(\omega) = 2nK$$

K → Extinction coefficient
n, K → Two optical constt

★ These optical constt. can be related to reflectance & absorption coefficient.

Ex: At normal incidence.



★ **Then**

Reflectance = R → can be obtained by intensities of incident & reflected wave $= E_{ref}^* E_{ref} / E_{inc}^* E_{inc}$

R → correspond to, reflectivity $= r = \frac{E_{ref}}{E_{inc}}$

★

If components of $\vec{E}, \vec{B}$ are ∥ to crystal surface are continuous, Then

$$r = \frac{n + ik - 1}{n + ik + 1}$$

* Reflectance = $R = \dfrac{E^*_{ref}\,E_{ref}}{E^*_{inc}\,E_{inc}} = r^*.r = \dfrac{(n-1)^2+K^2}{(n+1)^2+K^2} \rightarrow$ eq Ⓐ

1st approach to find n, K

- eq Ⓐ is generalised to non-normal angles of incidence (say θ)
- Another expr. for reflectance is find at angle of θ incidence, in terms of θ, n(ω), K(ω) and polarisation of incident radiation.
- when we compare above eqns result with expt. measured reflectance at θ, then n(ω), K(ω) can be found.

2nd approach to find n, K

- Use of Kramers - Kronig relations -

$$n(\omega) = 1 + P\int_{-\infty}^{\infty} \frac{d\omega'}{\pi}\,\frac{K(\omega')}{\omega'-\omega}$$

$$K(\omega) = -P\int_{-\infty}^{\infty} \frac{d\omega'}{\pi}\,\frac{n(\omega')-1}{\omega'-\omega}$$

P → Principal Value integral